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Translation and scale invariants of Legendre moments

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Abstract

By convention, the translation and scale invariant functions of Legendre moments are achieved by using a combination of the corresponding invariants of geometric moments. They can also be accomplished by normalizing the translated and/or scaled images using complex or geometric moments. However, the derivation of these functions is not based on Legendre polynomials. This is mainly due to the fact that it is difficult to extract a common displacement or scale factor from Legendre polynomials. In this paper, we introduce a new set of translation and scale invariants of Legendre moments based on Legendre polynomials. The descriptors remain unchanged for translated, elongated, contracted and reflected non-symmetrical as well as symmetrical images. The problems associated with the vanishing of odd-order Legendre moments of symmetrical images are resolved. The performance of the proposed descriptors is experimentally confirmed using a set of binary English, Chinese and Latin characters. In addition to this, a comparison of computational speed between the proposed descriptors and the aforesaid geometric moments-based method is also presented.

Keywords: Legendre moments; Translation and scale invariants; Image normalization; Aspect ratio invariants; Geometric moments

1. Introduction

Moments with Legendre polynomials as kernel function, denoted as Legendre moments, were first introduced by Teague [1]. Legendre moments belong to the class of orthogonal moments, and they were used in several pattern recognition applications [2–4]. They can be used to attain a near zero value of redundancy measure in a set of moment functions, so that the moments correspond to independent characteristics of the image [5]. The definition of Legendre moments, however, has a form of projection of the image intensity function onto the Legendre polynomials. This coordinate representation does not easily yield translation and scale invariant functions because it is difficult to extract a common displacement or scale factor from Legendre polynomials. Because of this, to the authors' knowledge, no report has been published on two-dimensional translation and scale invariants of Legendre moments based on Legendre polynomials.

There are, however, two other moment approaches currently being used to obtain the translation and scale invariants of Legendre moments: (a) image normalization method (INM) and (b) indirect method (IDM) [6,7]. The former first computes the normalization parameters, and knowing them, Legendre moments of the standard image can be computed from the examined image. At present, the normalization is accomplished by using either geometric moments or complex moments. INM standardizes each shape by setting its zeroth order geometric or complex moments, m_{00} or C_{00}, respectively, to a predetermined value, β. Legendre moments of the standard image can then be computed from its centroid by redefining the image coordinate (x, y) of the

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scaled image such that [6]:

$$\dot{x} = \left(x - \frac{\dot{m}_{10}}{\dot{m}_{00}}\right)\sqrt{\frac{\beta}{\dot{m}_{00}}}; \quad \dot{y} = \left(y - \frac{\dot{m}_{01}}{\dot{m}_{00}}\right)\sqrt{\frac{\beta}{\dot{m}_{00}}}, \quad (1.1)$$

where geometric moments of orders $(p+q)$ of the transformed image is given by $\dot{m}_{pq} = a^{p+q+2}\int_x\int_y x^p y^q f(x,y)\,dx\,dy$. This scheme is only applicable for uniformly scaled images. For images with different scale factors, a and b, along x and y dimensions, respectively, there are two methods currently being used to redefine the image coordinate (x, y) for computing Legendre moments of the standard image [8,9]:

(a) Palaniappan's method:

$$\ddot{x} = \left(x - \frac{\ddot{m}_{10}}{\ddot{m}_{00}}\right)\sqrt{\frac{\ddot{m}_{20}}{\varepsilon_x.\ddot{m}_{00}}} \quad \text{and}$$

$$\ddot{y} = \left(y - \frac{\ddot{m}_{01}}{\ddot{m}_{00}}\right)\sqrt{\frac{\ddot{m}_{02}}{\varepsilon_y.\ddot{m}_{00}}}, \quad (1.2)$$

where $\varepsilon_x = \sqrt{m_{20}/m_{00}}$ and $\varepsilon_y = \sqrt{m_{02}/m_{00}}$ are predetermined values of the standard image, and

(b) Alghoniemy's method:

$$\ddot{x} = \left(x - \frac{\ddot{m}_{10}}{\ddot{m}_{00}}\right)\sqrt{\frac{\beta.\varepsilon}{\ddot{m}_{00}}}; \quad \ddot{y} = \left(y - \frac{\ddot{m}_{01}}{\ddot{m}_{00}}\right)\sqrt{\frac{\beta}{\varepsilon.\ddot{m}_{00}}}$$

$$\text{and} \quad \varepsilon = \frac{l_y}{l_x}, \quad (1.3)$$

where l_y and l_x denote the height and the width of $f(x, y)$, respectively. The two-dimensional geometric moments of order $(p+q)$ of original and non-uniformly scaled images are, respectively, given as follows:

$$m_{pq} = \int_x\int_y x^p y^q f(x,y)\,dx\,dy \quad \text{and}$$

$$\ddot{m}_{pq} = a^{p+1}b^{q+1}\int_x\int_y x^p y^q f(x,y)\,dx\,dy. \quad (1.4)$$

It is quite evident that the translation and scale invariance can be easily achieved by image normalization method regardless of the feature descriptors. Moments computed using this scheme, however, may differ from true moments of the standard image. This is because the normalization parameters may not always correspond to an exact transformation of the scaled image. This method works well if the image is expanded or contracted with an integer scale factor. Small error is introduced to the normalization parameter when the scaling factor is of a non-integer value. This example is illustrated in Fig. 1.

Indirect method, on the other hand, makes use of translation and scale invariants of geometric moments to form the corresponding invariants of Legendre moments. They are given as follows [7]:

$$\hat{L}_{pq} = \frac{(2p+1)(2q+1)}{4}\sum_{h=0}^{p}\sum_{l=0}^{q} B_{ph}B_{ql}M_{hl}, \quad (1.5)$$

where the translation invariants, μ_{pq}, and both translation and scale invariants of geometric moments, M_{pq}, are, respectively, defined as follows [10]:

$$\mu_{pq} = \sum_{e=0}^{p}\sum_{f=0}^{q} {}^pC_e\,{}^qC_f(-x_0)^e(-y_0)^f m_{(p-e)(q-f)} \text{ and}$$

$$M_{pq} = \frac{\mu_{pq}.\mu_{00}^{\xi+1}}{\mu_{(p+\xi)0}.\mu_{0(q+\xi)}}, \; \xi = 0, 1, 2, 3\ldots \quad (1.6)$$

and B_{pk} denotes Legendre polynomial coefficients defined in Eq. (3.5). Eqs. (1.5) and (1.6) can be used for translated, elongated, contracted and/or reflected images. This method calculates the monomial basis set of $\{x^p y^q\}$ of geometric moments within the image space. This considerably shortens the time taken to compute the invariants of Legendre moments as compared to that of computing Legendre polynomials within the image space. However, a fraction of time is allocated to compute the terms like B_{pk} and ${}^nC_r$. This allocation becomes significant to overall time when order p is large and approximates N. It may be necessary to develop a recurrence relation for the coefficients to avoid them from any factorial loops.

It can be observed that the aforesaid present methods achieve the translation and scale invariance indirectly via moments other than the original Legendre moments of the examined image. Their respective function is not directly derived from the translated and/or scaled Legendre polynomials. In this paper, we introduce a new set of translation and scale invariants of Legendre moments based on Legendre polynomials. The translation invariants are derived from Legendre central moments. They are developed for non-symmetrical as well as symmetrical images. The problems associated with the vanishing of odd-order Legendre moments of symmetrical images are addressed and resolved. The scale invariants are achieved by algebraically eliminating the scale factor contained in the scaled Legendre moments. They remain unchanged for elongated, contracted and reflected images. The proposed descriptors eliminate the requirement of using INM, or employing other moments in IDM to achieve the corresponding invariants of Legendre moments. Their performance is experimentally confirmed using a set of binary English, Chinese and Latin characters. In addition to this, a comparison of computational speed between the proposed descriptors and IDM is also presented.

The organization of this paper is given as follows: In Section 2, we show the theory of Legendre moments. It is followed by the derivation and performance analysis of the proposed translation invariants with symmetrical and non-symmetrical images in Section 3. Section 4 shows the derivation of the proposed scale descriptors. Also, this

	**Original** Standard Mass, β = **230**	**SCALE FACTOR: 3.0** Mass = **2070** **a** = $\sqrt{\frac{\text{Mass}}{\beta}}$ = **3.0** Deviation = **0%**	**SCALE FACTOR: 4.8** Mass = **5342** **a** = $\sqrt{\frac{\text{Mass}}{\beta}}$ = **4.8193** Deviation = **0.4%**
	(a)	(b)	(c)

Fig. 1. The accuracy of normalization parameter, a.

section presents a comparative study of invariant capability and computational speed between the proposed and present descriptors. Section 5 concludes the study. The invariant sets used in the numerical analysis are listed in Appendix A(i) and A(ii).

2. Legendre moments

The two-dimensional Legendre moments of order $(p+q)$, with image intensity function $f(x,y)$, are defined as [7]

$$L_{pq} = \frac{(2p+1)(2q+1)}{4} \int_{-1}^{1}\int_{-1}^{1} P_p(x) \times P_q(y) f(x,y)\,dx\,dy; \quad x,y \in [-1,1], \tag{2.1}$$

where Legendre polynomial, $P_p(x)$, of order p is given by

$$P_p(x) = \sum_{k=0}^{p} \left\{ (-1)^{\frac{p-k}{2}} \frac{1}{2^p} \frac{(p+k)!x^k}{\left(\frac{p-k}{2}\right)!\left(\frac{p+k}{2}\right)!k!} \right\}_{p-k=\text{even}}. \tag{2.2}$$

The recurrence relation of Legendre polynomials, $P_p(x)$, is given as follows:

$$P_p(x) = \frac{(2p-1)xP_{p-1}(x) - (p-1)P_{p-2}(x)}{p}, \tag{2.3}$$

where $P_0(x)=1, P_1(x)=x$ and $p>1$. Since the region of definition of Legendre polynomials is the interior of $[-1,1]$, a square image of $N \times N$ pixels with intensity function $f(i,j)$, $0 \leqslant i,j \leqslant (N-1)$, is scaled in the region of $-1 < x,y < 1$. In the result of this, Eq. (2.1) can now be expressed in discrete form as

$$L_{pq} = \lambda_{pq} \sum_{i=0}^{N-1}\sum_{j=0}^{N-1} P_p(x_i)P_q(y_j)f(i,j), \tag{2.4}$$

where the normalizing constant,

$$\lambda_{pq} = \frac{(2p+1)(2q+1)}{N^2}.$$

x_i and y_j denote the normalized pixel coordinates in the range of $[-1,1]$, which are given by

$$x_i = \frac{2i}{N-1} - 1 \quad \text{and} \quad y_j = \frac{2j}{N-1} - 1. \tag{2.5}$$

3. Translation invariants of Legendre moments

The effect resulted from the change of location of an image on moment computation can be cancelled out by considering its translation invariant property. The direct description of translation invariant of 2D Legendre moments can be obtained by evaluating their central moments, which is defined as follows:

$$\varphi_{pq} = \frac{(2p+1)(2q+1)}{4} \int_{-1}^{1}\int_{-1}^{1} P_p(x-x_0) \times P_q(y-y_0)\, f(x,y)\,dx\,dy, \tag{3.1}$$

where the centroids of x- and y-coordinate, x_0 and y_0, respectively, are derived as

$$x_0 = \frac{\sum\sum x f(x,y)}{\sum\sum f(x,y)} \quad \text{and} \quad y_0 = \frac{\sum\sum y f(x,y)}{\sum\sum f(x,y)}. \tag{3.2}$$

The intensity centroid can be expressed in terms of Legendre moments as

$$x_0 = \left(\frac{\lambda_{00}}{\lambda_{10}}\right)\left(\frac{L_{10}}{L_{00}}\right) \quad \text{and} \quad y_0 = \left(\frac{\lambda_{00}}{\lambda_{01}}\right)\left(\frac{L_{01}}{L_{00}}\right). \tag{3.3}$$

To derive $(p+q)$th-order Legendre central moments, the translated Legendre polynomials is first expressed as a series of decreasing powers of $(x-x_0)$:

$$\text{Translated}: P_p(x-x_0) = B_{pp}(x-x_0)^p + B_{p(p-2)}(x-x_0)^{p-2} + B_{p(p-4)}(x-x_0)^{p-4} + \cdots + B_{p\dot{k}}(x-x_0)^{\dot{k}}, \tag{3.4}$$

where $\dot{k}=1$ if p is odd, and 0 otherwise, and Legendre polynomial coefficient is defined as

$$B_{pk} = (-1)^{(p-k)/2} \frac{1}{2^p} \frac{(p+k)!}{((p-k)/2)!((p+k)/2)!k!}. \tag{3.5}$$

The individual polynomial expression in Eq. (3.4) is further expanded to get the following series of powers of x:

$$\begin{aligned} P_p(x-x_0) &= B_{pp}(x^p) + (-{}^pC_1x_0B_{pp})(x^{p-1}) \\ &\quad + (B_{p(p-2)} + {}^pC_2x_0^2B_{pp})(x^{p-2}) \\ &\quad + (-{}^{(p-2)}C_1x_0B_{p(p-2)} - {}^pC_3x_0^3B_{pp})(x^{p-3}) \\ &\quad + (B_{p(p-4)} + {}^{(p-2)}C_2x_0^2B_{p(p-2)} \\ &\quad + {}^pC_4x_0^4B_{pp})(x^{p-4}) + (-{}^{(p-4)}C_1x_0B_{p(p-4)} \\ &\quad - {}^{(p-2)}C_3x_0^3B_{p(p-2)} - {}^pC_5x_0^5B_{pp})(x^{p-5})\ldots \end{aligned} \tag{3.6}$$

We can then express the translated Legendre polynomials in terms of original Legendre polynomials by substituting the original Legendre polynomials into the series of decreasing powers of x in Eq. (3.6) as shown below:

$$P_p(x-x_0) = \sum_{n=0}^{p} \upsilon_{p(p-n)}P_{p-n}(x); \qquad n \in N, \tag{3.7}$$

where $\upsilon_{pp} = 1$ and

$$\begin{aligned} \upsilon_{p(p-n)} &= \tau_{p-n}\left[\sum_{r=1}^{n}(-x_0)^{r(p-n+r)}C_rB_{p(p-n+r)} \right. \\ &\quad \left. - \sum_{s=1}^{n-1}\upsilon_{p(p-s)}B_{(p-s)(p-n)}\right], \end{aligned} \tag{3.8}$$

$$\tau_{p-n} = \frac{1}{B_{(p-n)(p-n)}}; \qquad n-r = \text{even};$$

$$n-s = \text{even}; \qquad n \geqslant 1.$$

Likewise, the translated Legendre polynomials along y-direction can be deduced using the same procedures in Eqs. (3.6) and (3.7):

$$P_q(y-y_0) = \sum_{m=0}^{q} \kappa_{q(q-m)}P_{q-m}(y); \qquad m \in N, \tag{3.9}$$

where $\kappa_{qq} = 1$ and

$$\begin{aligned} \kappa_{q(q-m)} &= \tau_{(q-m)}\left[\sum_{u=1}^{m}(-y_0)^{u(q-m+u)}C_uB_{q(q-m+u)} \right. \\ &\quad \left. - \sum_{w=1}^{m-1}\kappa_{q(q-w)}B_{(q-w)(q-m)}\right], \end{aligned} \tag{3.10}$$

$$\tau_{(q-m)} = \frac{1}{B_{(q-m)(q-m)}}; \qquad m-u = \text{even};$$

$$m-w = \text{even}; \qquad m \geqslant 1.$$

Using the translated Legendre polynomials derived in Eqs. (3.7) and (3.9), the general form of two-dimensional Legendre central moments, in terms of Legendre moments, are subsequently obtained as follows:

$$\varphi_{pq} = \sum_{n=0}^{p}\sum_{m=0}^{q} \frac{\lambda_{pq}}{\lambda_{(p-n)(q-m)}}\upsilon_{p(p-n)}\kappa_{q(q-m)}L_{(p-n)(q-m)}. \tag{3.11}$$

Eq. (3.11) works well for non-symmetrical images. However, it gives zero values for odd order central moments when it is used for images with symmetry along x and/or y directions, and symmetry at centroid. This problem is due to the image centroid used in the calculation of central moments. Since a symmetrical image has pairs of pixels that are equidistant (in opposite directions) from its centroid, the sum of an odd order moment calculation produces a net value of zero. In computing the first 10 orders central moments, only 30% of them produce non-zero values. These limited features may cause difficulties in pattern classifications.

Therefore, the general form of two-dimensional Legendre central moments in Eq. (3.11) is modified by changing the coefficients $\upsilon_{p(p-n)}$ in Eq. (3.8) and $\kappa_{q(q-m)}$ in Eq. (3.10) such that their x_0 and y_0 are replaced by $(x_0 - x_s)$ and $(y_0 - y_s)$, respectively. This enables the central moments computed from a center other than the centroid of the translated image. The shift terms in x and y-axes, x_s and y_s, respectively, are calculated as follows:

$$x_s = \rho\left[\frac{L_{20}}{L_{00}} - 5x_0\frac{L_{10}}{L_{00}} + \left(\frac{15x_0^2}{2} + \frac{5}{2}\right)\right]^{1/2} \tag{3.12}$$

and

$$y_s = \varrho\left[\frac{L_{02}}{L_{00}} - 5y_0\frac{L_{01}}{L_{00}} + \left(\frac{15y_0^2}{2} + \frac{5}{2}\right)\right]^{1/2} \tag{3.13}$$

and the shift factors, ρ and ϱ can take any non-zero values. They are used to increase the discriminant power of moments of different orders. x_s and y_s are designed such that Eq. (3.11) preserves the translation invariance and gives non-zero values for all odd-orders of central moments of symmetrical and non-symmetrical images, it can also be used together with the scale descriptors proposed in Section 4.

3.1. Experimental results

In this section, two experiments are carried out to verify the proposed translation descriptors of Legendre moments when they are used for non-symmetrical as well as symmetrical images. In the first experiment, two 20×20-resolution non-symmetrical binary Latin and Chinese characters, as shown in Table 1, are used. They are shifted up, down, left and right as well as diagonally within an $N \times N$ image frame. Selected orders of central moments, as listed in Appendix A(i), are computed for each translation, and they are recorded in Table 1. Table 1 shows that the central moments for both images are non-zero, and they remain unchanged for all the translations. Moreover, Table 1 shows that the descriptors for Chinese character are easily differentiated from

Table 1
Selected orders of Legendre central moments for a binary latin and chinese characters

Image	Translation	φ_{20}	φ_{02}	φ_{11}	φ_{21}	φ_{12}	φ_{30}	φ_{03}
ÿ	$\Delta i = -1,\ \Delta j = -1$	-1.728E-01	-1.033E-01	-1.680E-02	-2.947E-03	-1.410E-02	3.328E-03	1.381E-02
	$\Delta i = -1,\ \Delta j = +1$	-1.728E-01	-1.033E-01	-1.680E-02	-2.947E-03	-1.410E-02	3.328E-03	1.381E-02
	$\Delta i = +1,\ \Delta j = -1$	-1.728E-01	-1.033E-01	-1.680E-02	-2.947E-03	-1.410E-02	3.328E-03	1.381E-02
	$\Delta i = +1,\ \Delta j = +1$	-1.728E-01	-1.033E-01	-1.680E-02	-2.947E-03	-1.410E-02	3.328E-03	1.381E-02
漪	$\Delta i = -1,\ \Delta j = -1$	-1.619E-01	-2.222E-01	-6.732E-03	2.598E-02	4.580E-03	-2.401E-02	-8.756E-03
	$\Delta i = -1,\ \Delta j = +1$	-1.619E-01	-2.222E-01	-6.732E-03	2.598E-02	4.580E-03	-2.401E-02	-8.756E-03
	$\Delta i = +1,\ \Delta j = -1$	-1.619E-01	-2.222E-01	-6.732E-03	2.598E-02	4.580E-03	-2.401E-02	-8.756E-03
	$\Delta i = +1,\ \Delta j = +1$	-1.619E-01	-2.222E-01	-6.732E-03	2.598E-02	4.580E-03	-2.401E-02	-8.756E-03

Table 2
Selected orders of Legendre central moments for a set of symmetrical binary characters and symbol without center relocation

Image	Symmetry	Translation	φ_{20}	φ_{02}	φ_{11}	φ_{21}	φ_{12}	φ_{30}	φ_{03}
H	[x and y-axes]	$\Delta i = -1, \Delta j = -1$	-0.33408	-0.41695	0.000	0.000	0.000	0.000	0.000
		$\Delta i = +1, \Delta j = +1$	-0.33408	-0.41695	0.000	0.000	0.000	0.000	0.000
E	x-axis	$\Delta i = -1, \Delta j = -1$	-0.49964	-0.25869	0.000	0.000	0.097721	0.126136	0.000
		$\Delta i = +1, \Delta j = +1$	-0.49964	-0.25869	0.000	0.000	0.097721	0.126136	0.000
U	y-axis	$\Delta i = -1, \Delta j = -1$	-0.33408	-0.30645	0.000	-0.1134	0.000	0.000	-0.1176
		$\Delta i = +1, \Delta j = +1$	-0.33408	-0.30645	0.000	-0.1134	0.000	0.000	-0.1176
□	centroid	$\Delta i = -1, \Delta j = -1$	-0.38973	-0.32539	0.000	0.000	0.000	0.000	0.000
		$\Delta i = +1, \Delta j = +1$	-0.38973	-0.32539	0.000	0.000	0.000	0.000	0.000

those of Latin character. These results validate the invariant and discriminative capabilities of the proposed descriptors.

In the second experiment, three 20×20-resolution binary English characters and one symbol with specific symmetrical properties, as shown in Table 2, are used. The characters comprise of **E**, **U** and **H** which are symmetrical along x-, y-axis and both x and y axes, respectively. The symbol used is □, which is symmetrical at centroid. The characters and symbol are shifted up, down, left, right and diagonally. The central moments, as listed in Appendix A(i), are computed for each translation. Table 2 shows that the central moments of image which is symmetrical along x-axis produce zero values when q is odd. Similarly, the central moments of image which is symmetrical along y-axis produce zero values when p is odd. In addition, the central moments of image which is symmetrical along both axes give zero values when p and/or q are odd. For image with symmetry at the origin, we find that zero value is obtained for central moments when $(p+q)$ is odd. It can be seen from Table 2 that 50% of the features used in this experiment produces zero values! The number of pattern features is significantly reduced. This may cause difficulties in pattern representation and classification.

To resolve this problem, the modified central moments, as defined in the previous section, are used. The second experiment is repeated to compute the selected orders of the modified central moments corresponding to those of Appendix A(i). Table 3 shows that modified central moments preserve the translation invariance achieved by Eq. (3.11), and they also maintain a non-zero value for all odd orders of central moments when they are applied to the corresponding symmetrical images in Table 2.

4. Scale invariants of Legendre moments

In object identification applications, the varying size of an image has impact on moment calculation. This is because image of an object is often produced at a different range from the image grabber. In order to have moments independent of size, a technique has to be developed to make them invariant to scale. Assuming that the original object is

Table 3
Selected orders of Legendre central moments for a set of symmetrical binary characters and symbol with center relocation

Image	Symmetry	Translation	φ_{20}	φ_{02}	φ_{11}	φ_{21}	φ_{12}	φ_{30}	φ_{03}
H	[x and y-axes]	$\Delta i = -1,\ \Delta j = -1$	-0.15383	-0.26157	0.20083	-0.10862	-0.19893	0.17739	0.01451
		$\Delta i = +1,\ \Delta j = +1$	-0.15383	-0.26157	0.20083	-0.10862	-0.19893	0.17739	0.01451
E	x-axis	$\Delta i = -1,\ \Delta j = -1$	-0.35660	-0.04336	0.21060	-0.29005	0.06898	0.03119	0.38654
		$\Delta i = +1,\ \Delta j = +1$	-0.35660	-0.04336	0.21060	-0.29005	0.06898	0.03119	0.38654
U	y-axis	$\Delta i = -1,\ \Delta j = -1$	-0.15383	-0.11792	0.22121	-0.23306	-0.08968	0.17739	0.11897
		$\Delta i = +1,\ \Delta j = +1$	-0.15383	-0.11792	0.22121	-0.23306	-0.08968	0.17739	0.11897
□	centroid	$\Delta i = -1,\ \Delta j = -1$	-0.26981	-0.18616	0.15506	-0.19625	-0.12567	-0.05350	0.06247
		$\Delta i = +1,\ \Delta j = +1$	-0.26981	-0.18616	0.15506	-0.19625	-0.12567	-0.05350	0.06247

scaled non-uniformly with different factors, a and b, along x and y-axes, respectively, the scale Legendre moments can be defined as follows:

$$\dot{L}_{pq} = \frac{(2p+1)(2q+1)}{4} \int_{-1}^{1}\int_{-1}^{1} P_p(ax)P_q(by) \times f(x,y)\,\mathrm{d}x\,\mathrm{d}y; \quad (a \neq b) \in (R - \{0\}). \tag{4.1}$$

The scaled Legendre polynomials along x-axis are first analyzed. They can be expressed as a series of decreasing powers of x as follows:

$$\text{Scaled}: P_p(ax) = B_{pp}a^p x^p + B_{p(p-2)}a^{p-2}x^{p-2} + B_{p(p-4)}a^{p-4}x^{p-4} + B_{p(p-6)}a^{p-6}x^{p-6} + \cdots + B_{p\dot{k}}a^{\dot{k}}x^{\dot{k}}. \tag{4.2}$$

The relation between the original and scaled Legendre polynomials is then formed by rearranging Eq. (4.2) as shown below:

$$\sum_{k=0}^{p} \delta_{pk}P_k(ax) = a^p \sum_{k=0}^{p} \delta_{pk}P_k(x); \quad (p-k) = \text{even}, \tag{4.3}$$

where $\delta_{pp} = 1$ and

$$\delta_{pk} = \sum_{r=0}^{\dot{p}-2} \frac{-B_{(p-r)k}\delta_{p(p-r)}}{B_{kk}}; \quad \dot{p} - r = \text{even}; \quad \dot{p} = (p-k) \geqslant 2. \tag{4.4}$$

By employing Eqs. (4.3) and (4.4), the scaled Legendre polynomials along y-direction can then be deduced as follows:

$$\sum_{d=0}^{q} \delta_{qd}P_d(by) = b^q \sum_{d=0}^{q} \delta_{qd}P_d(y); \quad (q-d) = \text{even}. \tag{4.5}$$

Eqs. (4.3) and (4.5) form the kernel of the scale invariants of Legendre moments. The invariants are denoted as ψ_{pq}. They are expressed as a series of $(p+q)$th, $(p+q-2)$th, $(p+q-4)$th, etc. orders of original or scaled Legendre moments as follows:

$$\psi_{pq} = \sum_{k=0}^{p}\sum_{d=0}^{q}\left[\frac{\lambda_{pq}}{\lambda_{kd}}\delta_{pk}\delta_{qd}\dot{L}_{kd}\right] = a^{p+1}b^{q+1}\sum_{k=0}^{p}\sum_{d=0}^{q}\left[\frac{\lambda_{pq}}{\lambda_{kd}}\delta_{pk}\delta_{qd}L_{kd}\right]. \tag{4.6}$$

Using the following relations:

1. $\psi_{00} = abL_{00}$,
2. $\psi_{p0} = a^{p+1}b\sum_{k=0}^{p}\frac{\lambda_{p0}}{\lambda_{k0}}\delta_{pk}L_{k0}$,
3. $\psi_{0q} = ab^{q+1}\sum_{d=0}^{q}\frac{\lambda_{0q}}{\lambda_{0d}}\delta_{qd}L_{0d}$,

the scale factors, a and b contained in Eq. (4.6) can be cancelled out. The normalized scale invariants of Legendre moments, ω_{pq}, are subsequently derived as follows:

$$\omega_{pq} = \frac{\psi_{pq}\psi_{00}^{\xi+1}}{\psi_{(p+\xi)0}\psi_{0(q+\xi)}}; \quad p, q \text{ and } \xi = 0, 1, 2, 3, \ldots . \tag{4.7}$$

The descriptors derived in Eq. (4.7) are denoted as aspect ratio invariants. They are applicable to images with uniform as well as non-uniform scaling. If a and/or b are of negative values, then the equation is used for inverted or reflected images. The existence of the proposed descriptors eliminate the requirement of using INM, or other moments in IDM to achieve the scale invariance of Legendre moments. However, the computational speed of the proposed descriptors in Eqs. (3.11) and (4.7) may be affected by factorial functions in their respective coefficients namely $\upsilon_{p(p-n)}$ and $\kappa_{q(q-m)}$, and δ_{pk}. To resolve this, we propose the following recurrence relations for Legendre polynomial coefficients to avoid

Table 4
The proposed scale descriptors for a non-uniformly contracted, expanded and reflected chinese character

Image	Scale & Translation	Proposed Scale Descriptors						
		ω_{20}	ω_{11}	ω_{02}	ω_{30}	ω_{21}	ω_{12}	ω_{03}
	**Original**	2.972E-02	1.999E+01	3.062E-02	6.583E-03	2.118E-01	1.799E-01	3.684E-02
	a=0.5, b=1.0 Δi=+1, Δj=−1	2.972E-02	1.999E+01	3.062E-02	6.583E-03	2.118E-01	1.799E-01	3.684E-02
	a =1.0, b=0.5 Δi=+2, Δj=−2	2.972E-02	1.999E+01	3.062E-02	6.583E-03	2.118E-01	1.799E-01	3.684E-02
	a = 0.5 b = −1.0 Δi=+3, Δj=−3	2.972E-02	1.999E+01	3.062E-02	6.583E-03	2.118E-01	1.799E-01	3.684E-02
	a = 1.0 b = −0.5 Δi = +4 Δj = −4	2.972E-02	1.999E+01	3.062E-02	6.583E-03	2.118E-01	1.799E-01	3.684E-02
	a = −0.5 b = 1.0 Δi = +5 Δj = −5	2.972E-02	1.999E+01	3.062E-02	6.583E-03	2.118E-01	1.799E-01	3.684E-02
	a = −1.0 b = 0.5 Δi = +6 Δj = −6	2.972E-02	1.999E+01	3.062E-02	6.583E-03	2.118E-01	1.799E-01	3.684E-02
	a = −2.0 b = −0.5 Δi = +7 Δj = −7	2.972E-02	1.999E+01	3.062E-02	6.583E-03	2.118E-01	1.799E-01	3.684E-02
	a = −0.5 b = −2.0 Δi = +8 Δj = −8	2.972E-02	1.999E+01	3.062E-02	6.583E-03	2.118E-01	1.799E-01	3.684E-02
σ/μ %		**0.00**	**0.00**	**0.00**	**0.00**	**0.00**	**0.00**	**0.00**

Table 5
The proposed scale descriptors for a non-uniformly contracted, expanded and reflected arabic number

Image	Scale & Translation	Proposed Scale Descriptors						
		ω_{20}	ω_{11}	ω_{02}	ω_{30}	ω_{21}	ω_{12}	ω_{03}
	Original	2.295E-02	7.077E-01	2.185E-02	1.814E-02	4.926E-02	4.829E-02	1.514E-02
	a=0.5, b=1.0 Δi=+1, Δj=–1	2.295E-02	7.077E-01	2.185E-02	1.814E-02	4.926E-02	4.829E-02	1.514E-02
	a =1.0, b=0.5 Δi=+2, Δj=–2	2.295E-02	7.077E-01	2.185E-02	1.814E-02	4.926E-02	4.829E-02	1.514E-02
	a = 0.5 b = –1.0 Δi=+3, Δj=–3	2.295E-02	7.077E-01	2.185E-02	1.814E-02	4.926E-02	4.829E-02	1.514E-02
	a = 1.0 b = –0.5 Δi = +4 Δj = –4	2.295E-02	7.077E-01	2.185E-02	1.814E-02	4.926E-02	4.829E-02	1.514E-02
	a = –0.5 b = 1.0 Δi = +5 Δj = –5	2.295E-02	7.077E-01	2.185E-02	1.814E-02	4.926E-02	4.829E-02	1.514E-02
	a = –1.0 b = 0.5 Δi = +6 Δj = –6	2.295E-02	7.077E-01	2.185E-02	1.814E-02	4.926E-02	4.829E-02	1.514E-02
	a = –2.0 b = –0.5 Δi = +7 Δj = –7	2.295E-02	7.077E-01	2.185E-02	1.814E-02	4.926E-02	4.829E-02	1.514E-02
	a = –0.5 b = –2.0 Δi = +8 Δj = –8	2.295E-02	7.077E-01	2.185E-02	1.814E-02	4.926E-02	4.829E-02	1.514E-02
σ/μ%		**0.00**	**0.00**	**0.00**	**0.00**	**0.00**	**0.00**	**0.00**

Table 6
A comparison of the present and proposed scale descriptors for a non-uniformly scaled arabic number

Image	Method	Scaling	ω_{02}	ω_{03}	ω_{11}	ω_{12}	ω_{20}	ω_{21}	ω_{30}
5	Proposed	a=4.2, b=5.1	2.174E-02	1.516E-02	6.970E-01	4.746E-02	2.284E-02	4.934E-02	1.772E-02
		a=5.2, b=4.1	2.175E-02	1.517E-02	6.991E-01	4.767E-02	2.284E-02	4.933E-02	1.780E-02
		a=4.3, b=5.2	2.174E-02	1.503E-02	6.289E-01	4.550E-02	2.278E-02	4.890E-02	1.724E-02
		a=5.3, b=4.2	2.178E-02	1.511E-02	6.420E-01	4.565E-02	2.282E-02	4.936E-02	1.739E-02
		a=4.4, b=5.4	2.180E-02	1.539E-02	8.045E-01	5.103E-02	2.290E-02	5.109E-02	1.876E-02
								σ/μ %	**2.99**
	IDM	a=4.2, b=5.1	6.421E+00	9.850E+00	2.135E+02	3.270E+01	6.744E+00	3.400E+01	1.151E+01
		a=5.2, b=4.1	6.422E+00	9.855E+00	2.141E+02	3.285E+01	6.745E+00	3.399E+01	1.156E+01
		a=4.3, b=5.2	6.420E+00	9.766E+00	1.926E+02	3.135E+01	6.727E+00	3.370E+01	1.120E+01
		a=5.3, b=4.2	6.430E+00	9.819E+00	1.966E+02	3.146E+01	6.738E+00	3.401E+01	1.130E+01
		a=4.4, b=5.4	6.438E+00	1.000E+01	2.464E+02	3.516E+01	6.762E+00	3.521E+01	1.219E+01
								σ/μ %	**2.99**
	INM	a=4.2, b=5.1	-3.233E-03	9.384E-04	-4.470E-05	-5.530E-04	-3.255E-03	6.736E-04	-7.808E-04
		a=5.2, b=4.1	-3.172E-03	7.281E-04	-4.225E-05	-6.624E-04	-3.152E-03	5.167E-04	-9.247E-04
		a=4.3, b=5.2	-3.142E-03	8.904E-04	-4.230E-05	-5.373E-04	-3.161E-03	6.387E-04	-7.579E-04
		a=5.3, b=4.2	-3.128E-03	7.140E-04	-4.145E-05	-6.523E-04	-3.108E-03	5.066E-04	-9.106E-04
		a=4.4, b=5.4	-3.162E-03	9.157E-04	-4.342E-05	-5.362E-04	-3.184E-03	6.574E-04	-7.572E-04
								σ/μ %	**7.60**

IDM ≡ Geometric Moments-based Scale Invariants of Legendre Moments
INM ≡ Image Normalization Method using Geometric Moments

them from any factorial iterations:

$$B_{(p-2)k} = \frac{-(p-k)}{p+k-1} B_{pk}, \tag{4.8}$$

$$B_{p(k-2)} = \frac{-k(k-1)}{(p+k-1)(p-k+2)} B_{pk}. \tag{4.9}$$

These relations enable the entire set of polynomial coefficients, B_{pk}, of fixed order p or index k to be derived in a single loop of computation.

4.1. Experimental results

In this section, two experiments are carried out to verify the proposed scale invariants in the previous section. In the first experiment, the proposed descriptors in Eq. (4.7) are tested with translated, non-uniformly contracted or expanded, and reflected images. In the second experiment, a comparison of performance between the present and proposed scale descriptors is presented.

In the first experiment, a 20×20-resolution binary Chinese character and arabic number "5", as shown in Tables 4 and 5, are employed. They are arbitrarily displaced from the original position. They are then non-uniformly expanded or contracted, and inverted along x and y-axes. Selected orders of the proposed scale descriptors in Eq. (4.7) are computed for the respective transformed character and number. They are then recorded in Tables 4 and 5. Both tables show that the values of the descriptors remain unchanged for different non-uniform scaling and inversion. The deviation, σ/μ%, is zero. In addition to this, the descriptors in Table 4 can be easily differentiated from those of Table 5. These results verify the invariant and discriminative properties of the proposed scale descriptors.

In the second experiment, a comparative analysis of the performance of the present and proposed methods is presented. The first experiment is repeated for the present methods, namely INM and IDM, as stated in Eqs. (1.1) and (1.5), respectively. The binary arabic number "5" is non-uniformly expanded along x- and y-axes with different non-integer scale factors. Its descriptors are recorded in Table 6. It is clearly seen that the proposed descriptors present an equally well performance as those of IDM. INM, however, gives the largest deviation, due to the computed normalization

Table 7
The CPU elapsed time (s) used to compute scale descriptors of Legendre moments

Image	Scale Factor	Proposed Method			IDM		
		0 to 24	0 to 36	0 to 48	0 to 24	0 to 36	0 to 48
	1	0.03	0.09	0.22	0.10	0.35	1.05
	2	0.04	0.11	0.24	0.11	0.37	1.12
	3	0.05	0.13	0.30	0.12	0.44	1.24
	4	0.07	0.18	0.38	0.18	0.55	1.43
	5	0.10	0.24	0.50	0.24	0.69	1.66

parameter does not correspond to the exact transformation of the image. The proposed descriptors may act as another alternative to the present method.

Meanwhile, in the same experiment, a comparative study of the computational speed between the proposed descriptors and IDM is also presented. A 20×20-resolution binary Chinese character, as shown in Table 7, is expanded with a uniform scale factor, $a = b$, from 1 to 5 along x- and y-axes. This increases the image size, N with a stepsize of a^2. The time taken to compute the entire set of the proposed descriptors from 0 to 24 orders, 0 to 36 orders and 0 to 48 orders are collected and tabulated in Table 7. The corresponding time taken by IDM are also listed in Table 7. It can be seen from Table 7 that the proposed descriptors take a shorter time than those of IDM when order p and/or image size N increases. The proposed method consumes 0.10, 0.24 and 0.50 s to compute the respective maximum of 24, 36 and 48 orders for 100×100-resolution image. In the corresponding case, IDM takes 0.24, 0.69 and 1.66 s, which are approximately 2–3 times of computational time than those of taken by the proposed method. The performance of both methods, however, still has room to be improved. The algorithms proposed by researchers in Refs. [11–14] can be used to increase the speed of IDM as well as the proposed method.

5. Concluding remarks

In this paper, we have shown a new alternative to achieve translation and scale invariance of Legendre moments. We have presented a mathematical framework to derive a new set of translation and scale invariants of Legendre moments based on Legendre polynomials. The translation invariants are derived from Legendre central moments. They have been developed for non-symmetrical as well as symmetrical images. The scale invariants, on the other hand, are achieved by algebraically eliminating the scale factor contained in the scaled Legendre moments. They remain unchanged for elongated, contracted and reflected images. The results of simulation have demonstrated the invariant and discriminative capabilities of the proposed descriptors. The results have also shown that the direct computation of the proposed invariants takes shorter time than that of the indirect method.

Appendix A

(i) Translation invariants of Legendre moments

Order 1:

$$\varphi_{10} = L_{10} - x_0 \left[\frac{\lambda_{10}}{\lambda_{00}}\right] L_{00},$$

$$\varphi_{01} = L_{01} - y_0 \left[\frac{\lambda_{01}}{\lambda_{00}}\right] L_{00}.$$

Order 2:

$$\varphi_{20} = L_{20} - 3x_0 \left[\frac{\lambda_{20}}{\lambda_{10}}\right] L_{10} + \frac{3x_0^2}{2} \left[\frac{\lambda_{20}}{\lambda_{00}}\right] L_{00},$$

$$\varphi_{02} = L_{02} - 3y_0 \left[\frac{\lambda_{02}}{\lambda_{01}}\right] L_{01} + \frac{3y_0^2}{2} \left[\frac{\lambda_{02}}{\lambda_{00}}\right] L_{00},$$

$$\varphi_{11} = L_{11} - x_0 \left[\frac{\lambda_{11}}{\lambda_{01}}\right] L_{01} - y_0 \left[\frac{\lambda_{11}}{\lambda_{10}}\right] L_{10} + x_0 y_0 \left[\frac{\lambda_{11}}{\lambda_{00}}\right] L_{00}.$$

Order 3:

$$\varphi_{30} = L_{30} - 5x_0 \left[\frac{\lambda_{30}}{\lambda_{20}}\right] L_{20} + \frac{15x_0^2}{2} \left[\frac{\lambda_{30}}{\lambda_{10}}\right] L_{10} - \left[x_0 + \frac{5x_0^3}{2}\right] \left[\frac{\lambda_{30}}{\lambda_{00}}\right] L_{00},$$

$$\varphi_{03} = L_{03} - 5y_0 \left[\frac{\lambda_{03}}{\lambda_{02}}\right] L_{02} + \frac{15y_0^2}{2} \left[\frac{\lambda_{03}}{\lambda_{01}}\right] L_{01} - \left[y_0 + \frac{5y_0^3}{2}\right] \left[\frac{\lambda_{03}}{\lambda_{00}}\right] L_{00},$$

$$\varphi_{21} = L_{21} - y_0 \left[\frac{\lambda_{21}}{\lambda_{20}}\right] L_{20} - 3x_0 \left[\frac{\lambda_{21}}{\lambda_{11}}\right] L_{11} + 3x_0 y_0 \left[\frac{\lambda_{21}}{\lambda_{10}}\right] L_{10} + \frac{3x_0^2}{2} \left[\frac{\lambda_{21}}{\lambda_{01}}\right] L_{01} - \frac{3x_0^2 y_0}{2} \left[\frac{\lambda_{21}}{\lambda_{00}}\right] L_{00},$$

$$\varphi_{12} = L_{12} - x_0 \left[\frac{\lambda_{12}}{\lambda_{02}}\right] L_{02} - 3y_0 \left[\frac{\lambda_{12}}{\lambda_{11}}\right] L_{11} + 3x_0 y_0 \left[\frac{\lambda_{12}}{\lambda_{01}}\right] L_{01} + \frac{3y_0^2}{2} \left[\frac{\lambda_{12}}{\lambda_{10}}\right] L_{10} - \frac{3x_0 y_0^2}{2} \left[\frac{\lambda_{12}}{\lambda_{00}}\right] L_{00}.$$

(ii) Aspect ratio invariants of Legendre moments

Order 1:

$$\omega_{10} = \frac{\psi_{10}\psi_{00}^3}{\psi_{30}\psi_{02}}; \quad \omega_{01} = \frac{\psi_{01}\psi_{00}^3}{\psi_{20}\psi_{03}}.$$

Order 2:

$$\omega_{20} = \frac{\psi_{20}\psi_{00}^3}{\psi_{40}\psi_{02}}; \quad \omega_{02} = \frac{\psi_{02}\psi_{00}^3}{\psi_{20}\psi_{04}}; \quad \omega_{11} = \frac{\psi_{11}\psi_{00}^3}{\psi_{30}\psi_{03}}.$$

Order 3:

$$\omega_{30} = \frac{\psi_{30}\psi_{00}^3}{\psi_{50}\psi_{02}}; \quad \omega_{03} = \frac{\psi_{03}\psi_{00}^3}{\psi_{20}\psi_{05}};$$

$$\omega_{21} = \frac{\psi_{21}\psi_{00}^3}{\psi_{40}\psi_{03}}; \quad \omega_{12} = \frac{\psi_{12}\psi_{00}^3}{\psi_{30}\psi_{04}}.$$

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