Maths and Stats for Stage 1 & 2 COSC

A past student's guide

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E.J. Richardson

Contents

1 Introduction

The purpose of this booklet is to provide COSC students with useful formula, explanations and information about basic maths and stats. As some COSC students take only stats at uni and others take only maths, the COSC department recognises that the mathematical and statistical strengths of different students will lie in different areas. With any luck this booklet should help students to overcome any areas of weakness that they have, or simply remind them of things they forgot the second they left school.

This document is not perfect and is certainly not the last word on maths and stats for COSC. If you have any corrections, questions or suggestions please send them to `ejr33@student.canterbury.ac.nz`.

Other Mathematics and Statistics resources that may be helpful include:

- Lambourne, R. & Tinker, M. (2000), *Basic Mathematics for the Physical Sciences.* Call Number: QA 37.2.B311 2000 (Physical Sciences Library)
- McClave, J., Dietrich, F. & Sincich, T. (2000), *Statistics,* 8^{th} *Ed.* Call Number: QA 276.12.M126 (Physical Sciences Library)
- Haeussler, E. & Paul, R. (1999), *Introductory Mathematical Analysis: for business, economics and the life and social sciences,* 9^{th} *Ed.* Call Number: QA 300.H137 1999 (Physical Sciences Library).
- Molloy, M. (1989), *Fundamentals of Performance Modeling,* Call Number: QA 76.9.C65.M727 (Physical Sciences Library).

2 General

2.1 Set Notation

$a \in X$	a is an element of the set X
$a \notin X$	a is not an element of X
$X \ni a$	X contains a
$X = \{a, b\}$	The elements a and b form the entire contents of the set X
$Y \subset X$	Y is a subset of X
$Y \not\subset X$	Y is not a subset of X
$Y = X$	Y and X are equal, that is, they contain identical elements
$Y \subseteq X$	Y is a subset of, or is equal to X
$Y \not\subseteq X$	Y is not a subset of, and is not equal to X
$X \supset Y$	X is the superset of Y
$X \not\supset Y$	X is not the superset of Y
$X \supseteq Y$	X is the superset of, or is equal to Y
$X \not\supseteq Y$	X is not the superset of, and is not equal to Y
$X \cup Y$	X union Y, that is, all of the elements of X and all of the elements of Y
$X \cap Y$	X intersection Y, that is, only those elements which are in both X and Y
$X \times Y$	The Cartesian (cross) product of X and Y, that is, every element in X paired with every element in Y
$\overline{X}$	The complement of X
$\{\}$ or $\emptyset$	The empty set
Ω	The universal set

2.2 Greek Letters

Greek letters are commonly used by mathematicians, statisticians and computer geeks. Unfortunately, these letters are not always written and pronounced in the same ways by different people. To further complicate matters some of these letters have come to have common meanings. However, it is worth knowing that some people will use Greek letters which have common meanings to mean something other than that which is widely accepted.

The two tables below list the symbols, names and common pronunciations of the letters of the Greek alphabet. In addition to these tables there is a further table which details the specific meanings which are associated with some of the Greek letters.

Lowercase Letters

α	alpha	(al-fuh)	ν	nu	(noo)
β	beta	(be-tuh)	ξ	xi	(ks-eye)
γ	gamma	(gam-uh)	o	omicron	(om-i-kron)
δ	delta	(del-tuh)	π	pi	(pie)
ϵ, ε	epsilon	(ep-sil-on)	ρ	rho	(row)
ζ	zeta	(ze-tuh)	σ, ς	sigma	(sig-muh)
η	eta	(ee-tuh)	τ	tau	(tau)
θ, ϑ	theta	(thee-tuh)	υ	upsilon	(oop-si-lon)
ι	iota	(eye-oh-tuh)	ϕ, φ	phi	(fee)
κ	kappa	(kap-uh)	χ	chi	(k-eye)
λ	lambda	(lam-duh)	ψ	psi	(sigh)
μ	mu	(myou)	ω	omega	(oh-me-guh)

Uppercase Letters

A	alpha	(al-fuh)	N	nu	(noo)
B	beta	(be-tuh)	Ξ	xi	(ks-eye)
Γ	gamma	(gam-uh)	O	omicron	(om-i-kron)
Δ	delta	(del-tuh)	Π	pi	(pie)
E	epsilon	(ep-sil-on)	P	rho	(row)
Z	zeta	(ze-tuh)	Σ	sigma	(sig-muh)
H	eta	(ee-tuh)	T	tau	(tau)
Θ	theta	(thee-tuh)	Υ	upsilon	(oop-si-lon)
I	iota	(eye-oh-tuh)	Φ	phi	(fee)
K	kappa	(kap-uh)	X	chi	(k-eye)
Λ	lambda	(lam-duh)	Ψ	psi	(sigh)
M	mu	(myou)	Ω	omega	(oh-me-guh)

Some common meanings of Greek letters

δ	Differences
Δ	Change
ε, ϵ	The empty set, or a very small number
μ	Population mean
π	Ratio of a circle's circumference to its diameter ($\pi \approx 3.14159$)
σ	Standard Deviation
Σ	Sum
Ω	Universal Set

2.3 Other Useful Symbols and Notations

$\exists a$	Existential quantifier, asserts existence, that is, a exists.
$\forall a$	Universal quantifier, asserts that a property holds for every member of a set, that is, all a's share some property which does not vary.
$\{\}$	Braces
$()$	Parentheses
$[]$	Brackets
∞	Infinity
$\therefore$	Therefore
$\sum$	Sum
$\approx$	Approximately equal
$\equiv$	Equivalent
$\neg a$	Boolean negation, that is, not a.
iff	If and only if

2.4 Binary Numbers

Binary numbers are used by computers to represent information. Binary numbers contain only 2 digits, 0 and 1. Just as the decimal system uses base 10, the binary system uses base 2. This means that the rightmost digit in a binary number represents 2^0, the next rightmost digit represents 2^1, the next represents 2^2, and so on. The diagram below shows how the digits '1111' represent very different numbers in the binary system (base 2) and in the decimal system (base 10). In the binary system the digits '1111' represent the number fifteen, which is made up of 1 lot of 8 (2^3), 1 lot of 4 (2^2), 1 lot of 2 (2^1) and 1 lot of 1 (2^0). In the decimal system the digits '1111' represent the number one thousand, one hundred and eleven, which is made up of 1 lot of 1000 (10^3), 1 lot of 100 (10^2), 1 lot of 10 (10^1) and 1 lot of 1 (10^0).

	Binary System				Decimal System			
Base	$\leftarrow 2^3$	2^2	2^1	2^0	$\leftarrow 10^3$	10^2	10^1	10^0
Number	1	1	1	1	1	1	1	1

In the binary system, 101 represents the number five. Expanded, the binary number 101 is equivalent to $2^0 + (0 \times 2^1) + 2^2$, or five. In order to explicitly state that a number is in binary, a subscript '2' may be placed after the number (e.g. 1011_2). The table below shows the first eleven binary numbers.

Binary	Decimal
0	0
1	1
10	2
11	3
100	4
101	5
110	6
111	7
1000	8
1001	9
1010	10

Table 1: Binary and Decimal numbers

2.4.1 Addition

Binary numbers never include any digits that are greater than 1. This is true because for any column, the column directly to the left of it is two times it's size. For example 2^1 (2) is two times the size of 2^0 (1).

When adding in decimal, $5 + 6 = 11$, 11_{10} can be thought of as one pile of 10, and 1 pile of 1. When adding in binary $1 + 1 + 1 = 11$, 11_2 can be thought of as one pile of 2 and one pile of 1. In binary, two ones in the same column (e.g. $1 + 1$) are equivalent to a single 1 in the column to the left of that column (e.g. 10). Below are some example which illustrate how to add binary numbers.

1. $$\begin{array}{r} 101+ \\ 1001 \\ \hline 1110 \end{array}$$

2. $$\begin{array}{r} 10101+ \\ 1101 \\ \hline 100010 \end{array}$$

3. $$\begin{array}{r} 1000010111+ \\ 10110111 \\ \hline 1011001110 \end{array}$$

4. $$\begin{array}{r} 1000001100100+ \\ 100111111100 \\ 110110110110 \\ \hline 10100000010110 \end{array}$$

2.5 Hexadecimal Numbers

Hexadecimal is the base 16 number system. Hexadecimal has 16 digits. Table 2 contains the 16 digits used in the base 16 system and their equivalent decimal values.

Hexadecimal	Decimal
0	0
1	1
2	2
3	3
4	4
5	5
6	6
7	7
8	8
9	9
A	10
B	11
C	12
D	13
E	14
F	15

Table 2: Hexadecimal and Decimal numbers

The rightmost column in a hexadecimal number represents 16^0, the next rightmost column represents 16^1, the next represents 16^2, and so on. The hexadecimal number C3 is equivalent to $(12 \times 16^1) + (3 \times 16^0)$ or the decimal number 195. In order to explicitly state that a number is in hexadecimal a subscript '16' may be placed after the number (e.g. $2D1F_{16}$).

2.6 Logarithms

The logarithmic function with base b, where $b > 0$ and $b \neq 1$, is written $\log_b$ and is defined by

$$y = \log_b x \quad \textit{if and only if} \quad b^y = x$$

Each logarithmic function is called the inverse of its exponential function. For example, $\log_2 16 = 4$ is the inverse of $2^4 = 16$. For this example, we would say that '4 is the log in base 2 of 16' which means that '2 raised to the power of four (2^4) is 16'. In this example, $\log_2 16 = 4$ is the logarithmic form and $2^4 = 16$ is the exponential form.
It is relatively easy to convert between logarithmic and exponential forms as

$$y = \log_b x \quad \text{means} \quad b^y = x$$

Below are some examples of converting between logarithmic and exponential forms.

$$\begin{array}{rrcllrcl}
1. & 6^2 & = & 36, & \text{therefore} & \log_6 36 & = & 2 \\
2. & 4^3 & = & 64, & \text{therefore} & \log_4 64 & = & 3 \\
3. & 10^0 & = & 1, & \text{therefore} & \log_{10} 1 & = & 0 \\
4. & \log_7 49 & = & 2, & \text{means} & 7^2 & = & 49 \\
5. & \log_{36} 6 & = & \frac{1}{2}, & \text{means} & 36^{\frac{1}{2}} & = & 6 \\
6. & \log_2 \frac{1}{16} & = & -4, & \text{means} & 2^{-4} & = & \frac{1}{16}
\end{array}$$

Logarithms that have the same base may be subtracted from, or added to each other. The two rules below govern how this may be done.

$$\log_b m + \log_b n = \log_b(mn)$$

$$\log_b m - \log_b n = \log_b \frac{m}{n}$$

Below are some examples of adding and subtracting logarithms.

$$
\begin{array}{lrcrcl}
1. & \log_{10} 3 & + & \log_{10} 7 & = & \log_{10}(3 \times 7) \\
 & & & & = & \log_{10}(21) \\
2. & \log_2 5 & + & \log_2 9 & = & \log_2(5 \times 9) \\
 & & & & = & \log_2(45) \\
3. & \log_8 \dfrac{1}{2} & + & \log_8 4 & = & \log_8(\frac{1}{2} \times 4) \\
 & & & & = & \log_8(2) \\
4. & \log_{10} 12 & - & \log_{10} 6 & = & \log_{10} \frac{12}{6} \\
 & & & & = & \log_{10} 2 \\
5. & \log_5 4 & - & \log_5 8 & = & \log_5 \frac{4}{8} \\
 & & & & = & \log_5 0.5 \\
6. & \log_6 17 & - & \log_6 12 & = & \log_6 \frac{17}{12} \\
 & & & & = & \log_6 1.4167
\end{array}
$$

It is possible to use your calculator (which will usually only have $\log_{10}$ & $\log_e$) to find the answer to logarithmic functions with any base using the following rule.

$$\log_b m = \frac{\log_a m}{\log_a b}$$

The examples below illustrate how this rule may be applied.

$$
\begin{array}{llcl}
1. & \log_2 8 & = & \dfrac{\log_{10} 8}{\log_{10} 2} \\
 & & = & 3
\end{array}
\qquad
\begin{array}{llcl}
2. & \log_5 25 & = & \dfrac{\log_{10} 25}{\log_{10} 5} \\
 & & = & 2
\end{array}
$$

$$
\begin{array}{llcl}
3. & \log_{16} 9 & = & \dfrac{\log_{10} 9}{\log_{10} 16} \\
 & & = & 0.7925
\end{array}
\qquad
\begin{array}{llcl}
4. & \log_8 26 & = & \dfrac{\log_{10} 26}{\log_{10} 8} \\
 & & = & 1.5668
\end{array}
$$

If you want to establish a variable which is an exponent, the logarithm rule below is very useful.

$$\log_b m^r = r$$

Below are two examples which illustrate how this rule may be applied.

$$1. \quad \log_{10} x^2 \quad = \quad 2 \qquad\qquad 2. \quad \log_{10} a^b \quad = \quad b$$

All of the Logarithm rules are listed below.

$$\log_b x = y \quad \text{where} \quad b^y = x \qquad\qquad \log_b 1 = 0$$

$$\log_b(mn) = \log_b m + \log_b n \qquad\qquad \log_b b^r = r$$

$$\log_b \frac{m}{n} = \log_b m - \log_b n \qquad\qquad b^{\log_b m} = m$$

$$\log_b m = \frac{\log_a m}{\log_a b} \qquad\qquad \log_b b = 1$$

$$\log_b m^r = r \log_b m \qquad\qquad \log(\tfrac{1}{x}) = -\log x$$

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3.1 Exponents

An exponent can also be called a power or an index. Exponents are computed after parentheses, but before division, multiplication, addition and subtraction.

4^3 is read 'four to the power of three' and is equivalent to $4 \times 4 \times 4$. In this example, 4 is the base, and 3 is the power or exponent.

'a to the power of 2' is written a^2 and is equivalent to $a \times a$. In this example, a is the base and 2 is the power or exponent.

a^b is read 'a to the power of b'. If $b = 4$, a^b is equivalent to $a \times a \times a \times a$. In this example a is the base and b is the power or exponent.

There are seven rules which can be used to simplify exponents, these are listed below.

1. $a^m \times a^n = a^{m+n}$ e.g. $2^2 \times 2^3 = 2^{2+3} = 2^5 = 32$

$$2^2 \times 2^3 = 2^{2+3} = 2^5 = 32$$

2. $a^m \div a^n = \frac{a^m}{a^n} = a^{m-n}$ e.g.

$$2^4 \div 2^2 = \frac{2^4}{2^2} = 2^{4-2} = 2^2 = 4$$

3. $(a^m)^n = a^{mn}$ e.g.

$$(2^2)^3 = 2^6 = 64$$

4. $(ab)^m = a^m b^m$ e.g.

$$(2a)^2 = 2^2 \times a^2 = 4a^2$$

e.g.

$$(3ab)^4 = 3^4 \times a^4 \times b^4 = 81a^4b^4$$

5. $a^0 = 1$ e.g. $2^0 = 1$

Exception: $0^0 = 0$

6. $\frac{1}{a^m} = a^{-m}$ e.g.

$$\frac{1}{2^3} = 2^{-3} = 0.125$$

7. $\sqrt{a^m} = a^{\frac{m}{2}}$ e.g. $\sqrt{a^8} = a^{\frac{8}{2}}$

e.g.

$$\sqrt{2^4} = 2^{\frac{4}{2}} = 2^2 = 4$$

It is important to note that rules 1 & 2 can only be applied to exponents that have the same base, that is, a must equal a. Problems with differing bases (such as $2^3 \times 4^2$) must be simplified without the assistance of the above rules.

3.2 Changing Between Bases

In order to change numbers between bases it is essential that we have some method of indicating which base a number is in. The base of a number may be indicated in subscript immediately after the number. For example, the number 101_2 is in the base 2 number system (binary), the number 101_{10} is in the base 10 number system (decimal), and the number 101_{16} is in the base 16 number system (hexadecimal).

3.2.1 Converting any base to decimal

To convert a number from any base system to decimal, you multiply each digit by the value of its column and then sum the results. That is, to find the value of 2145_a in decimal you would compute $(2 \times a^3) + (1 \times a^2) + (4 \times a^1) + (5 \times a^0)$. Below are some examples of how to convert numbers in different bases to decimal.

1. Convert 1011_2 to decimal
   $= (1 \times 2^3) + (0 \times 2^2) + (1 \times 2^1) + (1 \times 2^0)$
   $= 8 + 0 + 2 + 1$
   $= 11_{10}$

2. Convert 1220_3 to decimal
   $= (1 \times 3^3) + (2 \times 3^2) + (2 \times 3^1) + (0 \times 3^0)$
   $= 27 + 18 + 6 + 0$
   $= 51_{10}$

3. Convert 3231_4 to decimal
   $= (3 \times 4^3) + (2 \times 4^2) + (3 \times 4^1) + (1 \times 4^0)$
   $= 192 + 32 + 12 + 1$
   $= 237_{10}$

4. Convert 7523_8 to decimal
   $= (7 \times 8^3) + (5 \times 8^2) + (2 \times 8^1) + (3 \times 8^0)$
   $= 3584 + 320 + 16 + 3$
   $= 3923_{10}$

5. Convert 1086_9 to decimal
   $= (1 \times 9^3) + (0 \times 9^2) + (8 \times 9^1) + (6 \times 9^0)$
   $= 729 + 0 + 72 + 6$
   $= 807_{10}$

6. Convert $3FA8_{16}$ to decimal
   $= (3 \times 16^3) + (15 \times 16^2) + (10 \times 16^1) + (8 \times 16^0)$
   $= 12288 + 3840 + 160 + 8$
   $= 16296_{10}$

3.2.2 Converting decimal to other bases

To convert a number from decimal to any base system, first you must find the highest column value, in the new base, that is less than or equal to the number.

E.g. Convert 7_{10} to binary.

- The column 2^2 (4) is less than 7
- The column 2^3 (8) is greater than 7
- Therefore the column 2^2 is the column with the highest column value that is either less than or equal to 7.

Next, divide the number by that column value, to find the digit that should be placed in that column.

E.g. Convert 7_{10} to binary, continued.

- $7 \div 2^2$
- $7 \div 4 = 1$, remainder 3
- Therefore, the digit 1 should be placed in the column 2^2.

The remaining columns (those which contain lower column values) must be used to represent any remaining value of the number, after it has been divided by the highest column value. In order to find which digit should be placed in which columns the remainder (from dividing the highest column value by the number) should be divided by the second highest column value, then any remainder from this division should be divided by the next highest column value, and so on, until all of the columns have been assigned digits.

E.g. Convert 7_{10} to binary continued.

- $3 \div 2^1$
- $3 \div 2 = 1$, remainder 1
- Therefore, the digit 1 should be placed in the column 2^1.
- $1 \div 2^0$
- $1 \div 1 = 1$, no remainder
- Therefore, the digit 1 should be placed in the column 2^0.
- Hence we know that $7_{10} = 111_2$.

Note: if the remainder is 0, any columns which have not been assigned a digit will be assigned 0, as $0 \div x$ will always equal 0 for any possible value of x. Further examples of converting from decimal to other bases may be viewed below.

1. Convert 5_{10} to base 2
   2^2 is the highest column value that is ≤ 5
   $5 \div 2^2$
   $= 5 \div 4$
   $= 1$ (in the column 2^2), remainder 1
   $1 \div 2^1$
   $= 1 \div 2$
   $= 0$ (in the column 2^1), remainder 1
   $1 \div 2^0$
   $= 1 \div 1$
   $= 1$ (in the column 2^0)
   $5_{10} = 101_2$

2. Convert 320_{10} to base 16
   16^2 is the highest column value that is ≤ 320
   $320 \div 16^2$
   $= 320 \div 256$
   $= 1$ (in the column 16^2), remainder 64
   $64 \div 16^1$
   $= 64 \div 16$
   $= 4$ (in the column 16^1), remainder 0
   $0 \div 16^0$
   $= 0 \div 1$
   $= 0$ (in the column 16^0), remainder 0
   $320_{10} = 140_{16}$

3. Convert 271_{10} to base 8
   8^2 is the highest column value that is ≤ 271
   $271 \div 8^2$
   $= 271 \div 64$
   $= 4$ (in the column 8^2), remainder 15
   $15 \div 8^1$
   $= 15 \div 8$
   $= 1$ (in the column 8^1), remainder 7
   $7 \div 8^0$
   $= 7 \div 1$
   $= 7$ (in the column 8^0), remainder 0
   $271_{10} = 417_8$

To check your results after converting a number from decimal to another base, simply convert the number in the new base back to decimal.

3.3 Summation

Sigma Notation can be used to denote the sum of a sequence of numbers. Below are some examples of how sigma notation may be used.

$$\sum_{i=1}^{5} i \quad \text{is read `the sum of } i \text{ for } i = 1 \text{ to } i = 5\text{'.}$$

This is equivalent to $1 + 2 + 3 + 4 + 5$.

$$\sum_{i=1}^{n} x^i \quad \text{is read `the sum of } x^i \text{ for } i = 1 \text{ to } i = n\text{'.}$$

This is equivalent to $x^1 + x^2 + x^3 + x^4 + \ldots + x^n$.

$$\sum_{i=1}^{n} i^{i-1} \quad \text{is read `the sum of } i^{i-1} \text{ for } i = 1 \text{ to } i = n\text{'.}$$

This is equivalent to $1^0 + 2^1 + 3^2 + 4^3 + \ldots + n^{n-1}$.

Five useful summations are listed below.

$\sum_{i=1}^{n} k$	$=$	$k + k + k + \ldots + k$	$=$	nk		(constant)
$\sum_{i=1}^{n} i$	$=$	$1 + 2 + 3 + \ldots + n$	$=$	$\frac{n(n+1)}{2}$		(linear)
$\sum_{i=1}^{n} i^2$	$=$	$1 + 4 + 9 + \ldots + n^2$	$=$	$\frac{n(n+1)(2n+1)}{6}$		(quadratic)
$\sum_{i=1}^{n} 2^i$	$=$	$2 + 4 + 8 + \ldots + 2^n$	$=$	$2^{n+1} - 1$		(exponential)
$\sum_{i=1}^{n} \frac{1}{2^i}$	$=$	$\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \ldots + \frac{1}{2^n}$	$=$	$1 - \frac{1}{2^n}$		(inverse exponential)

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4.1 Matrices

Within this section matrix multiplication, and some formulae relevant to 2D graphics are covered. For more information on complex matrix manipulation see *Introductory Mathematical Analysis: for business, economics and the life and social sciences*, 9^{th} *Ed.* (as mentioned in Introduction).

4.1.1 Matrix Multiplication

Matrix Multiplication is the process of multiplying one matrix by another. Although matrix multiplication may seem tricky at first it is actually quite straight forward.

In order to be able to multiply one matrix by another (e.g. $A \times B$) the number of columns contained in the first matrix (A) must be equal to the number of rows contained in the second matrix (B).

To multiply one matrix by another you simply multiply the first element in the first *row* of the first matrix by the first element in the first *column* of the second matrix, and multiply the second element in the first row of the first matrix by the second element in the first column of the second matrix, and so on, until you multiply the nth element in the first row of the first matrix by the nth element in the first column of the second matrix. The results from these multiplications are added together and placed in the first element of the first row of the answer matrix. To find the second element of the first row of the answer matrix the first row of the first matrix is multiplied by the second column of the second matrix in the same way as the first row of the first matrix was multiplied by the first column of the second matrix. To obtain the first element of the second row of the answer matrix the second row of the first matrix is multiplied by the first column of the second matrix.

Below are two detailed examples of how to conduct matrix multiplication.

Example 1.

$$\begin{bmatrix} 2 & 4 \\ 6 & 8 \end{bmatrix} \begin{bmatrix} 1 & 3 \\ 5 & 7 \end{bmatrix} = \begin{bmatrix} ? & ? \\ ? & ? \end{bmatrix}$$

$$\begin{bmatrix} 2 & 4 \\ 6 & 8 \end{bmatrix} \begin{bmatrix} 1 & 3 \\ 5 & 7 \end{bmatrix} = \begin{bmatrix} (2 \times 1) + (4 \times 5) & ? \\ ? & ? \end{bmatrix}$$

$$\begin{bmatrix} 2 & 4 \\ 6 & 8 \end{bmatrix} \begin{bmatrix} 1 & 3 \\ 5 & 7 \end{bmatrix} = \begin{bmatrix} (2 \times 1) + (4 \times 5) & (2 \times 3) + (4 \times 7) \\ ? & ? \end{bmatrix}$$

$$\begin{bmatrix} 2 & 4 \\ 6 & 8 \end{bmatrix} \begin{bmatrix} 1 & 3 \\ 5 & 7 \end{bmatrix} = \begin{bmatrix} (2 \times 1) + (4 \times 5) & (2 \times 3) + (4 \times 7) \\ (6 \times 1) + (8 \times 5) & ? \end{bmatrix}$$

$$\begin{bmatrix} 2 & 4 \\ 6 & 8 \end{bmatrix} \begin{bmatrix} 1 & 3 \\ 5 & 7 \end{bmatrix} = \begin{bmatrix} (2 \times 1) + (4 \times 5) & (2 \times 3) + (4 \times 7) \\ (6 \times 1) + (8 \times 5) & (6 \times 3) + (8 \times 7) \end{bmatrix}$$

$$\begin{bmatrix} 2 & 4 \\ 6 & 8 \end{bmatrix} \begin{bmatrix} 1 & 3 \\ 5 & 7 \end{bmatrix} = \begin{bmatrix} 2 + 20 & 6 + 28 \\ 6 + 40 & 18 + 56 \end{bmatrix}$$

$$\begin{bmatrix} 2 & 4 \\ 6 & 8 \end{bmatrix} \begin{bmatrix} 1 & 3 \\ 5 & 7 \end{bmatrix} = \begin{bmatrix} 22 & 34 \\ 46 & 74 \end{bmatrix}$$

Example 2.

$$\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix} \begin{bmatrix} a & 1 \\ b & a \\ 0 & 2 \end{bmatrix} = \begin{bmatrix} ? & ? \\ ? & ? \end{bmatrix}$$

$$\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix} \begin{bmatrix} a & 1 \\ b & a \\ 0 & 2 \end{bmatrix} = \begin{bmatrix} 1a + 2b + (3 \times 0) & 1 + 2a + 6 \\ 4a + 5b + (6 \times 0) & 4 + 5a + 12 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix} \begin{bmatrix} a & 1 \\ b & a \\ 0 & 2 \end{bmatrix} = \begin{bmatrix} 1a + 2b & 7 + 2a \\ 4a + 5b & 16 + 5a \end{bmatrix}$$

4.1.2 Matrix Formulae

Below are the Matrix formulae which may be used to translate, scale and rotate 2D objects.

$$\text{2D Translation} = \begin{bmatrix} x' \\ y' \\ 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & T_x \\ 0 & 1 & T_y \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$

Note: T is the distance over which an axis will be translated.

$$\text{2D Scaling} = \begin{bmatrix} x' \\ y' \\ 1 \end{bmatrix} = \begin{bmatrix} S_x & 0 & 0 \\ 0 & S_y & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$

Note: S is the scaling factor

$$\text{2D Rotation} = \begin{bmatrix} x' \\ y' \\ 1 \end{bmatrix} = \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$

Note: θ is the angle through which the object is to be rotated.

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5.1 Chi Square Test for Goodness of Fit

In computer science the primary purpose of a Chi Square (χ^2) test for goodness of fit is to determine whether or not a sample group fits a given distribution. For example you may use the Chi Square test for goodness of fit to conclude whether or not a random number generator generates numbers which have a uniform distribution. In other subjects the primary purpose of a Chi Square test for goodness of fit may be to determine that a sample is *different* from a given distribution, this should not be confused with determining whether a sample has the *same* distribution as a given distribution.

In order to use the Chi Square test for goodness of fit to determine whether a sample of numbers fits a given distribution you must follow the steps below.

1. Divide the sample into a number of intervals (N).

   For example, if you had generated 1000 random numbers from 1 to 100, you could split your sample into 10 intervals. If you were comparing your sample to a uniform distribution, your first interval would contain all of the random numbers between 1 & 10, your second interval would contain all of the random numbers between 11 & 20 and so on.

2. Calculate the Chi Square test statistic using the following formula:

$$\chi^2 = \sum_{n=1}^{N} \frac{(O_n - E_n)^2}{E_n}$$

   In this formula $_n$ denotes the current interval, N denotes the total number of intervals. O is referred to as the Observed frequency and denotes the number of items in the sample which occurred in the current interval. E is referred to as the Expected frequency and denotes the number of items in the sample which we expected to occur in the current interval.

   For example, we generated 1000 random numbers from 1 to 100 which we expected to be uniformly distributed. Below are the observed and expected frequencies of our random numbers for each of our ten intervals.

n	Interval	Observed	Expected
1	1–10	110	100
2	11–20	98	100
3	21–30	99	100
4	31–40	103	100
5	41–50	106	100
6	51–60	90	100
7	61–70	97	100
8	71–80	101	100
9	81–90	97	100
10	91–100	99	100

The Chi Square statistic for our sample equals:

$$\chi^2 = \sum_{n=1}^{10} \frac{(O_n - E_n)^2}{E_n}$$

n	$O - E$	$(O - E)^2$	$\frac{(O-E)^2}{E}$
1	$110 - 100 = 10$	100	1.00
2	$98 - 100 = -2$	4	0.04
3	$99 - 100 = -1$	1	0.01
4	$103 - 100 = 3$	9	0.09
5	$106 - 100 = 6$	36	0.36
6	$90 - 100 = -10$	100	1.00
7	$97 - 100 = -3$	9	0.09
8	$101 - 100 = 1$	1	0.01
9	$97 - 100 = -3$	9	0.09
10	$99 - 100 = -1$	1	0.01
		$\chi^2 =$	2.70

3. Calculate the number of degrees of freedom (*df*) that your sample has. The number of degrees of freedom is equal to the number of intervals minus one.

   For example, our sample has 10 intervals, so it has 9 *df*.

4. Finally, you must compare your calculated χ^2 statistic with the critical values for Chi Square which can be viewed in Appendix A. Make sure that you look at the row which has the same number of degrees of freedom as your sample.
   If your Chi Square statistic is smaller than the Chi Square statistic in the row headed '$\chi^2_{\alpha=0.90}$' then you can be 90 percent sure that your sample has the same distribution as the distribution you have been comparing it to. If your Chi Square statistic is smaller than the Chi

Square statistic in the row headed '$\chi^2_{\alpha=0.95}$' then you can be 95 percent sure that your sample has the same distribution as the distribution you have been comparing it to. If your Chi Square statistic is smaller than the Chi Square statistic in the row headed '$\chi^2_{\alpha=0.99}$' then you can be 99 percent sure that your sample has the same distribution as the distribution you have been comparing it to. If your sample Chi Square statistic is bigger than all of the Chi square statistic in the row of the table with the same *df* then you cannot conclude how similar the distribution of your sample is to the distribution you compared it with.

For example, our Chi square statistic = 2.70 and our sample has 9 degrees of freedom. In the row in the table which has 9 *df* $\chi^2_{\alpha=0.90} = 4.16816$, $\chi^2_{\alpha=0.95} = 3.32511$ and $\chi^2_{\alpha=0.99} = 2.08791$. As our sample Chi square statistic is smaller than the statistics for both $\chi^2_{\alpha=0.90}$ and $\chi^2_{\alpha=0.95}$ but larger than the statistic for $\chi^2_{\alpha=0.99}$, therefore, we can be 95 percent sure (but not 99 percent sure) that our sample has a uniform distribution.

5.2 Completion of Squares

Completion of squares is simply a method which may be used to solve quadratic equations. Completing the square involves performing algebraic manipulations on the quadratic formula ($y = ax^2 + bx + c$). Namely, to complete the square you simply add and subtract $\frac{b^2}{4a}$ from the righthand side of the quadratic equation in the following way

$$y = \left(ax^2 + bx + \frac{b^2}{4a}\right) + c - \frac{b^2}{4a}$$

The reason for doing this is so that x may eventually be isolated. Below are two further algebraic manipulations that when performed on a quadratic equation which has had $\frac{b^2}{4a}$ added and subtracted from it will isolate x.

$$y = a\left(x^2 + \frac{b}{a}x + \frac{b^2}{4a^2}\right) + c - \frac{b^2}{4a}$$

$$y = a\left(x + \frac{b}{2a}\right)^2 + c - \frac{b^2}{4a}$$

Once x has been isolated in order to find the value of x you must simply rearrange the equation so that x is the only element on the left-hand side of the equals sign.

The example below illustrates how the completion of squares method may be used to solve a quadratic equation.

If $y = -0.01x^2 + 0.2x$ and $0 < x < 10$, find the value of x.

$$y = -0.01x^2 + 0.2x + 0 \qquad a = -0.01 \quad b = 0.2 \quad c = 0$$

$$\frac{b^2}{4a} \qquad = \frac{0.2^2}{4(-0.01)} \qquad = \frac{0.04}{4(-0.01)}$$

Next, we must add and subtract $\frac{b^2}{4a}$ from the right-hand side of the quadratic equation.

$$y = \left(-0.01x^2 + 0.2x + \frac{0.04}{4(-0.01)}\right) - \frac{0.04}{4(-0.01)}$$

Next, we perform the first step towards isolating x.

$$y = -0.01\left(x^2 + \frac{0.2}{-0.01}x + \frac{0.04}{4(-0.01)^2}\right) - \frac{0.04}{4(-0.01)}$$

The second step used to isolate x is performed next.

$$y = -0.01\left(x + \frac{0.2}{2(-0.01)}\right)^2 - \frac{0.04}{4(-0.01)}$$

We must now rearrange the equation so that x alone is on the left-hand side.

$$100y = -\left(x + \frac{0.2}{2(-0.01)}\right)^2 - \frac{4}{4(-0.01)}$$

$$100y = -\left(x + \frac{0.2}{-0.02}\right)^2 - \frac{4}{-0.04}$$

$$100y = -(x + -10)^2 - (-100) \qquad = -(x - 10)^2 + 100$$

$$100y - 100 = -(x - 10)^2 \qquad 0 = -(x - 10)^2 + 100 - 100y$$

$$(x-10)^2 = 100 - 100y \qquad = 100(1-y)$$

$$x - 10 = \pm\sqrt{100(1-y)} \qquad = \pm 10\sqrt{1-y}$$

$$x = \pm\left(10\sqrt{1-y}\right) + 10$$

As we know that $0 < x < 10$ we know that:

$$x = -\left(10\sqrt{1-y}\right) + 10$$

5.3 Probabilities

All probabilities refer to the likelihood of an event or events occurring. This likelihood is given a number between 0 & 1, with 0 meaning that the event will definitely not occur and 1 meaning that the event will certainly occur. Probabilities are either expressed as a decimal or a fraction but never as a percentage.

In any given situation there will be a certain number of events that may occur. Each of these events will have a certain probability (or likelihood) of occurring. The sum of the probabilities of all of the events will equal 1. For any situation the sum of the probabilities of all possible events will always equal 1, this is the total probability of the situation.

5.3.1 Conditional Probabilities

Conditional probabilities are used to evaluate how likely an event is to occur given that another event has already occurred. The formula which is used to compute conditional probabilities may be viewed below.

$$P(B \mid A) = \frac{P(A \cap B)}{P(A)}$$

Explained, this means that the probability of event B occurring given that event A has already occurred ($P(B \mid A)$), is equal to the probability of events A & B both occurring ($P(A \cap B)$) divided by the probability of event A occurring ($P(A)$).

For most problems you will encounter you will be given the probability of event A ($P(A)$) and the probability of event B ($P(B)$), but not the probability of both ($P(A \cap B)$). Provided that events A and B are independent, the probability of both events A and B, may be calculated using the following formula.

$$P(A \cap B) = P(A) \times P(B)$$

Below is an example of how conditional probabilities are calculated.

1. There are two fair coins, a 5 cent coin and a 20 cent coin, both of which are tossed. What is the probability that both coins land head up given that the 20 cent coin lands head up?

$$P(20Heads) = 0.5 \qquad P(5Heads) = 0.5$$

$$P(A \cap B) = P(A) \times P(B)$$

$$P(BothHeads) = P(20Heads) \times P(5Heads)$$

$$P(BothHeads) = 0.5 \times 0.5$$

$$P(B \mid A) = \frac{P(A \cap B)}{P(A)}$$

$$P(BothHeads \mid 20Heads) = \frac{P(BothHeads)}{P(20Heads)}$$

$$= \frac{0.5 \times 0.5}{0.5} \qquad = \frac{0.25}{0.5} \qquad = 0.5$$

5.4 Differentiation and Integration

5.4.1 Differentiation

Differentiation is used to find the slope (rate of change) of a curve at any given point. This means that for a given curve, we can use differentiation to provide us with a formula which we can then use to find the slope of that curve for any value of x. The formal definition of the derivative of a function (differentiation) is:

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

This is read 'f prime x' which equals the limit, as x tends to zero, of $\frac{f(x+h)-f(x)}{h}$. y is equal to $f(x)$ and the slope (rate of change) at any given point $= f'(x)$. In this formula, when the term $(x + h)$ is applied, every x, in $f(x)$, is simply replaced with $(x + h)$. Some examples of the application of this rule can be viewed below.

x	is replaced with	$(x + h)$
$x + 8$	is replaced with	$(x + h) + 8$
$4x$	is replaced with	$4(x + h)$
x^2	is replaced with	$(x + h)^2$
$6x^2$	is replaced with	$6(x + h)^2$
$x^3 + x$	is replaced with	$(x + h)^3 + (x + h)$
$5x^2 + 7x + 9$	is replaced with	$5(x + h)^2 + 7(x + h) + 9$

Below are two examples of how to differentiate using the formal definition of differentiation.

1. If $f(x) = x^2$, find the derivative of f.

$$f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} \qquad = \lim_{h\to 0} \frac{(x+h)^2 - x^2}{h}$$

$$= \lim_{h\to 0} \frac{x^2 + 2xh + h^2 - x^2}{h} \qquad = \lim_{h\to 0} \frac{2xh + h^2}{h}$$

$$= \lim_{h\to 0} 2xh + h \qquad \text{As } h \to 0 \text{ this } = 2x$$

Note: The simplest form must be found before $\lim_{h\to 0}$ is applied.

2. If $f(x) = 3x^2 + 2x + 4$, find the derivative of f.

$$f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h}$$

$$= \lim_{h\to 0} \frac{[3(x+h)^2 + 2(x+h) + 4] - (3x^2 + 2x + 4)}{h}$$

$$= \lim_{h\to 0} \frac{3x^2 + 3h^2 + 6xh + 2x + 2h + 4 - (3x^2 + 2x + 4)}{h}$$

$$= \lim_{h\to 0} \frac{3h^2 + 2h + 6xh}{h} \qquad = \lim_{h\to 0} 3h + 6x + 2 \qquad = 6x + 2$$

Unfortunately there are a number of different notations which are often used to denote $f'(x)$. Below is a list of these notations and their pronunciations.

Alternative Notations for $f'(x)$	
$\frac{dy}{dx}$	(dee-y, dee-x)
$\frac{d}{dx}[f(x)]$	(dee-f-x, dee-x)
y'	(y prime)
$D_x y$	(dee x of y)
$D_x[f(x)]$	(dee x of f-x)

Fortunately, there are a number of standard derivation rules (to find $f'(x)$ when given $f(x)$) which can be used in order to differentiate without using the formal definition above. Some of these rules can be found in the table below.

$f(x)$	$f'(x)$	$f(x)$	$f'(x)$
k (*constant*)	0	k^x	$(\log_e k)k^x$
kx^n	nkx^{n-1}	$\log_e(kx)$	$\frac{1}{x}$
$\log_k x$	$\frac{1}{x \log_e k}$		

Below are some examples of how the standard derivation rules may be applied.

1. If $f(x) = 5$ what does $f'(x)$ equal?
   When $f(x) = k,\ f'(x) = 0$, therefore, when $f(x) = 5,\ f'(x) = 0$

2. $f(x) = x^5$ what is the value of $f'(x)$?
   As $kx^n \ \rightarrow \ nkx^{n-1}, \quad f(x) = x^5 \ \rightarrow \ f'(x) = 5x^4$

3. If $f(x) = x^4 + 2x^2 + 2$ what does $f'(x)$ equal?
   $f'(x) = 4x^3 + (2 \times 2x^1) \quad = 4x^3 + 4x$

4. If $f(x) = 5^x + x^4$ what is the value of $f'(x)$?
   As $k^x \ \rightarrow \ (\log_e k)k^x \quad \& \quad kx^n \ \rightarrow \ nkx^{n-1}$,
   $f(x) = 5^x + +x^4 \quad \rightarrow \quad f'(x) = (\log_e 5)5^x + 4x^3$

Differentiation is a very large topic in Maths, for more information see *Introductory Mathematical Analysis: for business, economics and the life and social sciences,* 9^{th} *Ed.* or *Basic Mathematics for the Physical Sciences.* (as mentioned in Introduction).

5.4.2 Basic Integration

Integration is effectively the opposite of differentiation. For example, if differentiating function a gives us function b, then integrating function b will give us function a. However, it is important to note that if a function is differentiated and then integrated, the value of any constants contained in that function will be lost (that is, the value of any 'c''s from the formula $y = ax^2 + bx + c$ will be lost). Integration may be used to find the area between the curve of a graph and the x-axis.

The notation used to denote integration is

$$\int f(x)\,dx = F(x)$$

This means that integrating ($\int$) a function ($f(x)$) with respect to (d) a variable (x) equals a new function ($F(x)$). Below is a table which contains some rules which can be used to integrate basic functions, and an example which demonstrates the application of these rules.

$f(x)$	$F(x)$	$f(x)$	$F(x)$
k (a constant)	$kx + C$	$kx^n,\ n \neq -1$	$k\,\frac{x^{n+1}}{n+1} + C$
$k\,\frac{1}{x},\ x \neq 0$	$k \log_e \lvert x \rvert + C$	$\log_e(kx)$	$x \log_e(kx) - x + C$
$e^{\,kx}$	$\frac{1}{k}\,e^{\,kx} + C$		

Note: C is a constant

What does $\int 4x^3 + 12x^2 + 5\,dx$ equal?

As $k \quad \rightarrow \quad kx + c$

$$5 \quad \rightarrow \quad 5x + C$$

As $kx^n \quad \rightarrow \quad k\,\frac{x^{n+1}}{n+1} + C$

$$\begin{aligned} 4x^3 + 12x^2 \quad &\rightarrow \quad 4\,\frac{x^{3+1}}{4} + C + 12\,\frac{x^{2+1}}{2+1} + C \\ &\rightarrow \quad 4 \times \frac{1}{4}\,x^4 + C + 12 \times \frac{1}{3}\,x^3 + C \\ &\rightarrow \quad x^4 + C + 4x^3 + C \\ &\rightarrow \quad x^4 + 4^3 + C \end{aligned}$$

Therefore, $\int 4x^3 + 12x^2 + 5\,dx = x^4 + 4x^3 + 5x + C$

5.4.3 Integration Between Bounds

The purpose of integration between bounds is to find the area between the curve and the x-axis for a certain range. This range is specified by two x-values. Integration between bounds is denoted

$$\int_a^b f(x)dx = area$$

This means that we must integrate the function $f(x)$, between the bounds a and b, with respect to the variable x.

Three steps must be taken in order to integrate between bounds.

1. Integrate $f(x)$ with respect to x (exactly as we did in basic integration).
2. Write out the result of your integration twice, once with a substituted for x (Value A) and once with b substituted for x (Value B).
3. Subtract Value A from Value B.

Below are two examples of how to integrate between bounds.

1. What is the area under the curve $2x$ between $x = 1$ and $x = 3$?

$$\begin{aligned} area &= \textstyle\int_1^3 2x\ dx \\ &= \left[2\frac{x^2}{2} + C\right]_1^3 \quad = \quad \left[x^2 + C\right]_1^3 \\ &= (3^2 + C) - (1^2 + C) \quad = \quad 9 - 1 \\ &= 8 \end{aligned}$$

2. What is the area between the curve $4x^3 + 12x^2 + 5$, the x-axis, and the lines $x = -4$ and $x = 5$?

$$\begin{aligned}
area &= \int_{-4}^{5} 4x^3 + 12x^2 + 5\,dx \\
&= \left[x^4 + 4x^3 + 5x + C\right]_{-4}^{5} \\
&= (5^4 + 4(5^3) + 5(5) + C) - ((-4)^4 + 4(-4^3) + 5(-4) + C) \\
&= (625 + 4(125) + 25 + C) - (256 + 4(-64) + (-20) + C) \\
&= (625 + 500 + 25 + C) - (256 - 256 - 20 + C) \\
&= 1150 + C - (-20 + C) \\
&= 1150 + C + 20 - C \\
&= 1170
\end{aligned}$$

5.5 Permutations and Combinations

Permutations and Combinations are two different methods by which a subsets containing a certain number of elements may be selected from a larger set.

5.5.1 Permutations

Permutation is the grouping of elements in a way in which their order is important, that is $a, b, c, d \neq d, c, b, a$. For any set of N unique elements, from which you wish to select n elements in such a manner that their order is important, the number of different permutations is denoted by P_n^N and is equal to

$$\frac{N!}{(N-n)!}$$

Note: $N!$ is called N factorial and means $N \times (N-1) \times (N-2) \times \ldots \times 3 \times 2 \times 1$. Below are some examples of permutations.

1. Find the number of permutations of 5 elements taken 2 at a time.

$$P_2^5 = \frac{5!}{(5-2)!} = \frac{5 \times 4 \times 3 \times 2 \times 1}{3!} = \frac{120}{3 \times 2 \times 1} = \frac{120}{6} = 20$$

2. What is P_3^6?

$$P_3^6 = \frac{6!}{(6-3)!} = \frac{720}{3!} = \frac{720}{6} = 120$$

3. Find the number of permutations of 40 elements taken 2 at a time.

$$P_2^{40} = \frac{40!}{(40-2)!} = \frac{40!}{38!} = 1560$$

5.5.2 Combinations

Combination is the grouping of elements in a way in which their order is unimportant, that is $a, b, c, d \equiv d, c, b, a$. For any set of N elements from which you wish to select n elements the number of different samples (Combinations) of n elements is denoted by $\binom{N}{n}$ and is equal to

$$\frac{N!}{n!(N-n)!}$$

Below are some examples of combinations.

1. Find the number of combinations of 7 elements taken 4 at a time.

$$\binom{7}{4} = \frac{7!}{4!(7-4)!} = \frac{7\times 6\times 5\times 4\times 3\times 2\times 1}{4\times 3\times 2\times 1(3)!} = \frac{5040}{24(3\times 2\times 1)}$$
$$= \frac{5040}{24\times 6} = \frac{5040}{144} = 35$$

2. What is $\binom{56}{8}$?

$$\binom{56}{8} = \frac{56!}{8!(56-8)!} = 1420494075$$

A Critical Values of χ^2

df	$\chi^2_{\alpha=0.90}$	$\chi^2_{\alpha=0.95}$	$\chi^2_{\alpha=0.99}$
1	0.01580	0.00393	0.00016
2	0.21072	0.10259	0.02010
3	0.58438	0.35184	0.11483
4	1.06362	0.71072	0.29711
5	1.61031	1.14548	0.55430
6	2.20413	1.63539	0.87209
7	2.83311	2.16735	1.23904
8	3.48954	2.73264	1.64648
9	4.16816	3.32511	2.08791
10	4.86518	3.94030	2.55821
11	5.57779	4.57481	3.05347
12	6.30380	5.22603	3.57056
13	7.04150	5.89186	4.10691
14	7.78953	6.57063	4.66043
15	8.54675	7.26094	5.22935
16	9.31223	7.96164	5.81221
17	10.0852	8.67176	6.40776
18	10.8649	9.39046	7.01491
19	11.6509	10.1170	7.63273
20	12.4426	10.8508	8.26040
21	13.2396	11.5913	8.89720
22	14.0415	12.3380	9.54249
23	14.8479	13.0905	10.1957
24	15.6587	13.8484	10.8654
25	16.4734	14.6114	11.5240
26	17.2919	15.3791	12.1981
27	18.1138	16.1513	12.8786
28	18.9392	16.9279	13.5648
29	19.7677	17.7083	14.2565
30	20.5992	18.4926	14.9535
40	29.0505	26.5093	22.1643
50	37.6886	34.7642	29.7067
60	46.4589	43.1879	37.4848
70	55.3290	51.7393	45.4418
80	64.2778	60.3915	53.5400
90	73.2912	69.1260	61.7541
100	82.3581	77.9295	70.0648

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