Cantor’s Diagonalization Method

Let $N$ be the set of natural numbers, i.e., $N = \{1, 2, 3, \ldots\}$. Let $R$ be the set of real numbers between 0 and 1. If a set $S$ has a one-to-one correspondence with $N$, we say $S$ is countable. In other words, $S$ is denumerable, like $S = \{x_1, x_2, \ldots\}$. We also say we can enumerate $S$.

Theorem (Cantor). $R$ is not denumerable.

Proof. Let us assume $R$ is denumerable, that is, $R = \{x_1, x_2, \ldots\}$. Let the binary expansion of $x_i$ be given by $x_i = 0.b_{i1}b_{i2}b_{i3} \ldots$. The situation is illustrated in the following.

\[
\begin{align*}
x_1 & \quad 0.b_{11}b_{12}b_{13} \ldots \\
x_2 & \quad 0.b_{21}b_{22}b_{23} \ldots \\
\vdots & \\
x_i & \quad 0.b_{i1}b_{i2}b_{i3} \ldots \\
x_{i+1} & \quad 0.b_{i+1,1}b_{i+1,2}b_{i+1,3} \ldots \\
\vdots & 
\end{align*}
\]

Now let $x$ in $R$ be defined by $x = 0.b_{11}'b_{22}' \ldots b_{ii}' \ldots$, where $b'$ is the complement of $b$. We see that we cannot place $x$ anywhere in the above table. Suppose $x = x_i$. Then $x$ would disagree with $x_i$ at the $i$-th bit. In the above table $x$ will disagree at the diagonal points with any $x_i$ given by boldface.

Note. $R$ is denser than $N$. The density of $N$ is $\aleph_0$ (aleph zero), and that of $R$ is $\aleph_1$ (aleph one).