# Cantor's Diagonalization Method

Let N be the set of natural numbers, i.e., N = {1, 2, 3, …}. Let R be the set of real numbers between 0 and 1. If a set S has a one-to-one correspondence with N, we say S is countable. In other words, S is denumerable, like S = $\{x_1, x_2, \ldots\}$. We also say we can enumerate S.

Theorem (Cantor). R is not denumerable.
Proof. Let us assume R is denumerable, that is, R = $\{x_1, x_2, \ldots\}$. Let the binary expansion of $x_i$ is given by $x_i = 0.b_{i1}b_{i2}\ldots b_{ii}b_{i,i+1}\ldots$ The situation is illustrated in the following.

$x_1$ $0.\mathbf{b_{11}}b_{12}b_{13}\ldots$
$x_2$ $0.b_{21}\mathbf{b_{22}}b_{23}\ldots$
…
$x_i$ $0.b_{i1}\ldots$ $\mathbf{b_{ii}}b_{i,i+1}\ldots$
$x_{i+1}$ $0.b_{i+1,1}$ $\mathbf{b_{i+1,i+1}}b_{i+1,i+1}\ldots$
…

Now let x in R be defined by $x = 0.b_{11}'b_{22}'\ldots b_{ii}'\ldots$, where b' is the complement of b. We see that we cannot place x anywhere in the above table. Suppose $x = x_i$. Then x would disagree with $x_i$ at the i-th bit. In the above table x will disagree at the diagonal points with any $x_i$ given by boldface.

Note. R is denser than N. The density of N is $X_0$ (aleph zero), and that of R is $X_1$ (aleph one).