Chapter 1 Logic, Induction and Recursion

A statement or proposition is a sentence that is true or false.

Example.

(a) Ten is less than six.
(b) How are you?
(c) He is very talented.
(d) There are life forms outside the earth

(a) is false. (b) is not a proposition. (c) is not a proposition because "he" is a variable.
(d) is a proposition, whose truth value is not known yet.

Let A and B are propositions. We can define the new proposition A ^ B whose truth value is true when A and B are both true, and false otherwise. Similarly A v B is defined to be true when A or B is true, and false otherwise. These two operations are typical binary operations. For a unary operation, A' is true when A is false, and true otherwise. For two variables A and B, we have the following 16 propositions made from them. They are called logical functions with two variables. The table is called a truth table.

		(1)	(2)	(3)	(4)	(5)	(6)	(7)	(8)
A	B	A ^ B	A v B	A → B	A + B	A ↔ B	A' ^ B'	A' v B'	A' → B'
T	T	T	T	T	F	T	F	F	F
T	F	F	T	F	T	F	F	T	F
F	T	F	T	T	T	F	F	T	T
F	F	F	F	T	F	T	T	T	F

		(9)	(10)	(11)	(12)	(13)	(14)	(15)	(16)
A	B	A ← B	A' ← B'	A'	B'	A	B	F	T
T	T	T	F	F	F	T	T	F	T
T	F	T	T	F	T	T	F	F	T
F	T	F	F	T	F	F	T	F	T
F	F	T	F	T	T	F	F	F	T

(3) Logical implication. It reads "if A then B"
(4) Exclusive-or. This is true if either A or B is true.
(5) Logical coincidence. This is true if both of A and B are true or false.
(6) NOR. (A v B)' = A' ^ B' Comes from Not or
(7) NAND (A ^ B)' = A' v B' Comes from Not and
(8) Inverse implication
(9) Reverse implication
(10) Contraposition
(11) – (14) One variable function
(15) Logical constant F
(16) Logical constant T

Chapter 2 Sets and Combinatorics

Let A, B , C … denotes sets. $A \cap B$ is the intersection of A and B, which is the set of elements that belong to A and B. $A \cup B$ is the union of A and B, which is the set of elements that belong to A or B. If we talk about subsets of S, A' is the complement of A, which is the set of elements that do not belong to A (of course element of S).

Example $S = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$. $A = \{2, 3, 5, 7, 10\}$. $B = \{1, 4, 5, 7, 8, 10\}$.
$A' = \{1, 4, 6, 8, 9\}$. $A \cap B = \{5, 7, 10\}$. $A \cup B = \{1, 2, 3, 4, 5, 7, 8, 10\}$.

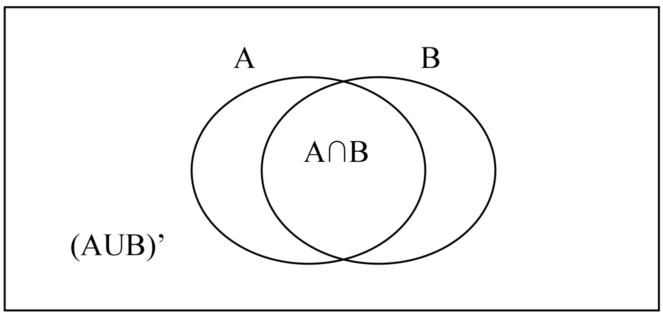
De Morgan' Law $(A \cup B)' = A' \cap B'$. $(A \cap B)' = A' \cup B'$

Example: From the above example. $A' = \{1, 4, 6, 8, 9\}$. $B' = \{2, 3, 6, 9\}$.

$(A \cap B)' = \{1, 2, 3, 4, 6, 8, 9\} = A' \cup B'$

Venn Diagram, The set operations are illustrated in a picture.

S



Cartesian product $A \times B = \{(x, y) \mid x \text{ is in } A \text{ and } y \text{ is in } B\}$. (x, y) ordered pair.

The number of elements in A is denoted by $|A|$, and is called he cardinality of A.

Inclusion and Exclusion principle
$|A'| = |S| - |A|$
$| A \cup B | = |A| + |B| - |A \cap B|$, $|A \cap B| = |A| + |B| - |A \cup B|$

The second $|A \cap B| = | ((A \cap B)')'| = |S| - | (A \cap B)'| =$
$|S| - | A' \cup B' | = |S| - (|A'| + |B'| - | A' \cap B'|) = |A| + |B| - |A \cup B|$

Example. The above example. $|A|=5$, $|B|=6$. $|A \cup B| = |A| + |B| - |A \cap B| = 11-3 = 8$

$|A \cup B \cup C| = |A| + |B| + |C| - |A \cap B| - |A \cap C| - |B \cap C| + |A \cap B \cap C|$

$|A \times B| = |A| \times |B|$. Example $|A \times B| = 30$.

Relations, Functions, and Matrices

Binary relation ρ between S and T is a subset of S x T.

Example S={1, 2, 3}, T={2, 3, 4}
If ρ is defined by “x=y”, ρ = {(2, 2), (3, 3)}
If ρ is defined by “x is smaller y”, ρ = {(1, 2), (,2 3), (3, 4)}

Exercise. List up ρ for “x≠y”, and “x+1=y”.

n-ary relation is a subset of “S_1 x S_2 x … x S_n”. A relational database if an n-ary relation.
A unary relation on S is a subset of S.

A binary relation on S is a subset of S x S = S^2.
Operations on binary relations on S, +, ·, ‘
x(ρ + σ)y ↔ xρy or xσy, x(ρ·σ)y ↔ xρy and xσy, xρ’y ↔ not xρy

Equivalence relation ρ on S if a
The following is satisfied for all x, y, and z

(1) Reflexive. X is in S → (x, x) is in ρ
(2) Symmetric. (x, y) is in ρ → (y, x) is in ρ
(3) Transitive. (x, y) is in ρ and (y, z) is in ρ → (x, z) is in ρ.

Example xρy ↔ x+y is even on {1, 2, 3}. ρ = {(1, 1), (2, 2), (3, 3), (1, 3), (3, 1)}
Exercise. List up equivalence relation “x+y is even” on S = {1, 2, 3, 4, 5}

Partial order

(1) Reflexive. X is in S → (x, x) is in ρ
(2) Anti-symmetric. (x, y) is in ρ and (y, x) is in ρ → x=y
(3) Transitive. (x, y) is in ρ and (y, z) is in ρ → (x, z) is in ρ.

Example. Let ρ be x≤y on {1, 2, 3}. ρ = {(1, 1), (2, 2), (3, 3), (1, 2), (2, 3), (1, 3)}
For power set of S, AρB is defined by A is a subset of B. If S={1, 2, 3},
{1, 2}ρ{1, 2, 3}, etc.

Equivalence relation divides the set S into equivalence classes
Example. x+y is even on {1, 2, 3, 4, 5}. We have two equivalence classes {1, 3, 5} and {2, 4}

Partial order has a Hasse diagram; if xρy there is an edge from x to y.
Example xρy is defined by x divides y on {1, 2, 3, 4, 6}.

```
      1
     / \
    2   3
   / \ /
  4   6
```

Edges are drawn upwards. They can be drawn the other way round.

Function f from S (domain) to T (co-domain) is a subset of S x T, that is, a kind of relation, that satisfies a special condition; if (x, y) is in f, x is in S, and if (x, y) is in f and (x, y') is in f, y=y'. That is for any x in S, some y corresponds in T, and many-to-one is possible, but one-to-many is not possible.
If (x, y) is in f, we write as y=f(x), and y is called the image of x under f.

Example. Function from S to T where S={1, 2, 3}, T={1, 2, 4, 9}$y=x^2$.
f={(1, 1), (2, 4), (3, 9)}

If y=f(x) and y=f(x') means x=x', f is one-to-one, or injection.

If for any y in T, there exists x in S such that y=f(x), f is said to be onto.

If f is one-to-one and onto, f is called a bijection from S to T.

Let f and g be functions from S to T and T to U. Composition of functions f and g, denoted by f o g, from S to U is defined by (fog)(x) = g(f(x)).

Matrices are m by n rectangular arrangements of numbers, called (m, n) matrices.

Example. $A = \begin{vmatrix} 1 & 2 & 3 \\ 2 & 3 & 4 \end{vmatrix}$ $B = \begin{vmatrix} 4 & 3 & 2 \\ 3 & 2 & 1 \end{vmatrix}$ $A + B = \begin{matrix} 5 & 5 & 5 \\ 5 & 5 & 5 \end{matrix}$

Let a_{ij} be the (i, j) elements of A. The matrix C = A + B is defined by its (I, j) element defined by $c_{ij} = a_{ij} + b_{ij}$. For (m, n) matrix A and (n, m) matrix B, the product of A and B, C=AB, is defined by

$$c_{ij} = \Sigma\ [k=1, \ldots, n]\ a_{ik}b_{kj}$$

Example $A = \begin{matrix} 1 & 2 \\ 3 & 4 \end{matrix}$ $B = \begin{matrix} 2 & 1 \\ 1 & 2 \end{matrix}$ $C = \begin{matrix} 4 & 5 \\ 10 & 11 \end{matrix}$

For (n n) square matrices, I is an identity matrix whose elements are all 0 except for diagonal elements, that is, (i, i) elements, which sre 1. Zero matrix O is the one whose elements are all 0.

We have IA = AI = A, A(B+C) = AB + AC, O+A = O, OA=AO=A, A-A=O, etc.
We can extend almost all properties on real numbers onto matrices.

The inverse matrix of A, A^{-1}, is defined by B such that AB=BA = I. The inverse exists if A is regular, that is, $|A| \neq 0$, where |A| is the determinant of A, defined by

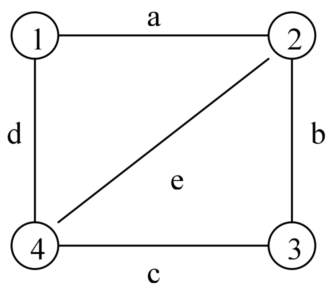
For $A = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}$. $|A| = a_{11}a_{22} - a_{12}a_{21}$, $A^{-1} = \begin{vmatrix} a_{22} & -a_{12} \\ -a_{21} & a_{11} \end{vmatrix} / |A|$

Example $A = \begin{matrix} 1 & 2 \\ 3 & 4 \end{matrix}$ $|A| = 4 - 6 \neq 0$, $A^{-1} = \begin{vmatrix} 4 & -2 \\ -3 & 1 \end{vmatrix} = \begin{vmatrix} -2 & 1 \\ 3/2 & -1/2 \end{vmatrix}$

Graphs and Trees

A graph is (N, A, f) where N is the set of node, A us the set of arcs, and f is a function from A to unordered pairs x-y of nodes x and y. A graph is given by a picture in which nodes correspond to points and arcs correspond to lines.

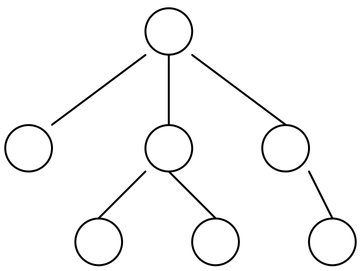
Example. N = {1, 2, 3, 4}, A = {a, b, c, d, e}. The correspondence f is denoted by :.
a: 1-2, b:2-3, c: 3-4, d:1-4, e:2-4



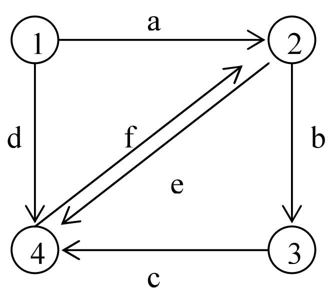
A path is a connected sequences of arcs, e.g., a, b, c is a path from 1 to 4. This is also given by a sequence of nodes, such as (1, 2, 3, 4). A simple path has no repetition of nodes. A cycle is a path from a node to itself. If a cycle has no repetition of nodes except for the starting, it is a simple cycle.

A tree is a connected graph with no cycle. A rooted tree is a tree one of whose nodes is specified as the root, often drawn at the top, and arcs are drawn downward.

Example.



The root has three children. The second child two children, etc.
A directed graph has a directed arc, that is, f is a function to ordered pairs (x, y).
In the following arcs e and f are difference.
.



a: (1, 2), b:(2, 3), c:(3, 4)
d:(1, 4), e:(2, 4), f:(4, 2)

Structures and Simulations

From logic operations and set operations we can extract the same structure.

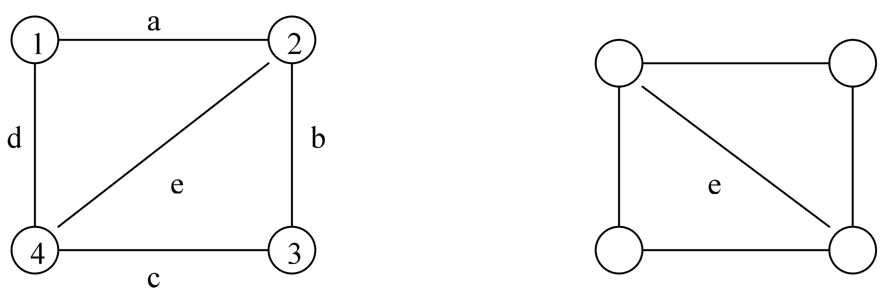
Logic	Sets
A v B = B v A commutative	A U B = B U A
A ^ (B ^ C) = (A ^ B) ^ C associative	A ∩ (B ∩ C) = (A ∩ B) ∩ C
…	…

From these example, we can extract an abstract structure, called Boolean algebra on set B, That satisfies the following rules.

1a x+y = y+x	1b xy = yx	(commutative)
2a x+(y+z) = x+(y+z)	2b (xy)z = x(yz)	(associative)
3a x+(yz) = (x+y)(x+z)	3b x(y+z) = xy + xz	(distributive)
4a x+0 = x	4b x1 = x	(identities)
5a x + x' = 1	5b xx' = 0	(complements)
6a x + x = x	6b xx = x	(idempotent)
7a x + xy = x	7b x(x+y) = x	(absorption)

This Boolean algebra simulates the operations on logic, sets and relations.

Another simulation: isomorphism and homomorphism



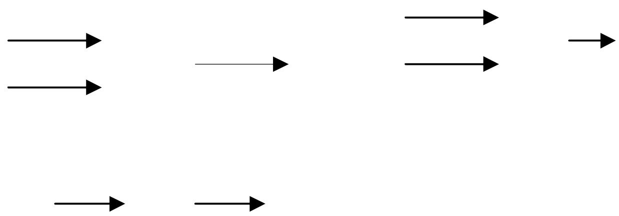
The two graphs are isomorphic under the correspondence (1 to 2), (2 to 3), (3 to 4), (4 to 1). The general definition is this. Let there be two graphs G=(N, A, f) and G'=(N', A' g). There is a bijection h from N to N' such that if there is an edge from between x and y in G, then there is one between h(x) and h(y).

For algebraic structures A=(S, *, a) and B=(T, &, b), where * and & are binary operations on S and T, and a and b symbolizes special elements in S and T (there can be more than one, but for simplicity we assume one for each). Let h be a function from S to T. If for any x and y, h(x*y) = h(x)&h(y), h(a)=b, h is said to be a homomorphism from A to B. If h is a bijection, it is said to be an isomorphism. If A and B are isomorphic, they are essentially the same structure.

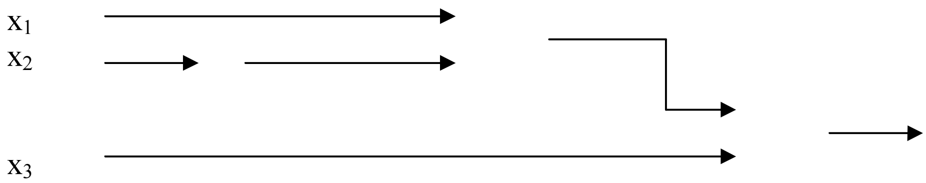
Example Let R be the set of real numbers. Let A=(R, +), B=(R^+, *).
Let h be defined by h(x)=2^x. Then h is a bijection, and we can verify, h(x+y) = 2^{x+y} = 2^x*2^y = h(x)*h(y). Thus A and B are isomorphic.
If h is a homomorphism from A to B, and not isomorphism, B is smaller than A, that is, B imitates the behavior of A in a smaller scale.

Boolean Algebra and Computer Logic

Boolean operations can be implemented by various physical methods, including electronic circuits. The signals of 1 and 0 are expressed by small voltages, such as +5mV and -5mV. Those logical operations are simulated by diodes. We simplify those operations by the following diagrams.



The Boolean function x1x2' + x3 is realized by the following Boolean network.
In this form, x1, x2', and x3 are called literals; x1, x3 positive, x2' negative
And we have three variables x1, x2, and x3.



Disjunctive normal form, or sum-of-products form

Pick up entries of f with 1. Make the sum of products, that correspond to those entries such that 1 corresponds to a positive literal and 0 corresponds to a negative literal.

x1	x2	x3	f(x1, x2, x3)
0	0	0	1
0	0	1	0
0	1	0	1
0	1	1	0
1	0	0	1
1	0	1	0
1	1	0	1
1	1	1	1

$f = x_1'x_2'x_3' + x_1'x_2x_3' + x_1x_2'x_3' + x_1x_2x_3' + x_1x_2x_3$
$= x_1'(x_2' + x_2)x_3' + x_1(x_2' + x_2)x_3' + x_1x_2x_3$
$= x_1'x_3' + x_1 x_3' + x_1x_2x_3$
$= x_3' + x_1x_2x_3$
$= x_3' + x_1x_2x_3' + x_1x_2x_3$ (absorption used)
$= x_3' + x_1x_2$ (simplified form)