COSC222 Tutorial 2  Predicate Logic and Induction

(1) Some additional inference rules for predicate logic

\( (\forall x)(P(x) \implies Q(x)) \)

\[ \implies (\forall x)P(x) \implies (\forall x)Q(x) \]

\( (\forall x)P(x) \)

\[ \equiv a \text{ is an arbitrary constant not in } P(x) \]

\( P(a) \)

\( (\exists x)P(x) \)

\[ \equiv t \text{ is a constant not used above this point} \]

\( P(t) \)

\( P(a) \)

\[ \equiv a \text{ is an arbitrary constant and } x \text{ does not appear in } P(a) \]

\( (\exists x)P(x) \)

\[ ((\exists x)P(x))' \equiv \text{ similar to de Morgan} \]

\( (\forall x)P'(x) \equiv \text{ similar to de Morgan} \]

\( (\forall x)P'(x) \)

\[ (((\forall x)P(x))')' \equiv \text{ similar to de Morgan} \]

\( A' \land B' \)

\( A' \lor B' \)

Using these rules, we prove the following.

Every microcomputer has a serial port. Some microcomputers have a parallel port. Therefore some microcomputers have a serial and a parallel port.

\( M(x) : x \text{ is a microcomputer} \)

\( S(x) : x \text{ has a serial port} \)

\( P(x) : x \text{ has a parallel port} \)

\[ (\forall x)(M(x) \implies S(x)) \]

\[ (\exists x)(M(x) \land P(x)) \]

\[ M(a) \implies S(a) \]

\[ M(a) \land P(a) \]

\[ M(a) \land P(a) \land S(a) \]

\[ (\exists x)(M(x) \land P(x) \land S(x)) \]

Following this example, prove the following. If one works hard, one gets rich. There are some poor people. Therefore we conclude that not everyone works hard. Use \( H(x) \) and \( R(x) \)
(2) A recurrence is given as follows:

\[
\begin{align*}
T(1) &= 0 \\
T(2n) &= 2T(n) + 2n - 1
\end{align*}
\]

Prove that \( T(n) = n\log_2 n - n + 1 \) for \( n = 2^m \) for some integer \( m \), that is, \( n \) is a power of 2.

Note. This is the number of comparisons used in mergesort, one of sorting algorithms more efficient than bubble-sort. We learn this algorithm in COSC229.

(3) A recurrence is given as follows:

\[
\begin{align*}
F(0) &= 0 \\
F(1) &= 1 \\
F(n) &= F(n-1) + F(n-2) \text{ for } n \geq 2
\end{align*}
\]

Prove that \( F(n) = \frac{1}{\sqrt{5}}(\frac{1+\sqrt{5}}{2})^n - \frac{1}{\sqrt{5}}(\frac{1-\sqrt{5}}{2})^n \) by induction.

Note. \( F(n) \) is the \( n \)-th Fibonacci number, 0, 1, 1, 2, 3, 5, 8, …

(4) Prove by induction that \( 3^n - 1 \) is divisible by 8 if \( n \) is even and positive.

(5) Define binomial coefficients \( C(n, k) \) by the recurrence

\[
\begin{align*}
C(n, 0) &= 1 \text{ for all } n \\
C(n, n) &= 1 \text{ for all } n \\
C(n+1, k) &= C(n, k) + C(n, k-1)
\end{align*}
\]

Prove that \( C(n, k) = \frac{n!}{k!(n-k)!} \) for \( 0 \leq k \leq n \) by induction on \( k \) and \( n \).