COSC413 Examination on Advanced Algorithms

16 June 1999, Room 121
Lecturer: Prof. T. Takaoka
Open book, calculators are allowed

(1) We judge whether *n* is a prime by the following algorithm, called the Eratosthenes sieve. This algorithm generate all primes up to *n*. Prepare an array *a*[2..*n*] and initialize all array elements to 0. Delete all numbers which are multiples of 2 from the candidates for primes. This is called the sieve by 2. Next delete all numbers which are multiples of 3. As 4 is already deleted, we come to 5. Delete all multiples of 5, etc, etc. If *i* is already deleted, we see *a*[*i*]=1, and we skip *i*. Each deletion operation is called a sieve. When we have finished the sieve by 5, the array looks like

i	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20
a[*i*]	0	0	1	0	1	0	1	1	1	0	1	0	1	1	1	0	1	0	1

(1.1) Write a program for the Eratosthenes sieve.

(1.2) Prove that we can stop at the the sieve by $\lfloor \sqrt{n} \rfloor$.

(1.3) Prove that the computing time is $O(n\log n)$.

Hint. Use the following approximation formula.

$$1 + 1/2 + 1/3 + ... + 1/n = O(\log n).$$

(2) Convert the following satisfiablity problem into the corresponding colorability problem, and discuss two corresponding solutions.

$$F = (x_1 + x_2)(x_1 + x_2' + x_3')(x_1' + x_2' + x_3)$$

Note. x’ is the negation of x, multiplication is ‘and’, and addition is ‘or’.

(3) Convert the above satisfiability problem into the corresponding clique problem, and discuss two corresponding solutions.

(4) Let a difference equation be defined by

$x(0) = 0,\ x(1) = 1$
$x(n+2) = 3x(n+1) - x(n)\ \ (n=0, 1, ...)$

(4.1) Obtain the solution for this equation.

(4.2) Transform the equation into a vector-matrix form as follows:

$$x(n+1) = 3x(n) - x(n-1)$$

This is transformed to the following.

$$(x(1), x(0)) = (1, 0)$$

$$(x(n+1), x(n)) = (x(n), x(n-1)) \begin{bmatrix} a11 & a12 \\ a21 & a22 \end{bmatrix} = (x(n), x(n-1))\, A$$

Obtain the matrix A. Then we have $(x(n+1), x(n)) = (x(1), x(0))\, A^n$. Obtain x(8) by repeated use of the equation 7 times and the repeated squaring of A three times, and confirm that you have the same result.

Note. Any linear recurrence of the above type is O(log*n*) computable.

(5) The Euclidean algorithm for greatest common divisors is given as follows:
For $a > b > 0$ such that $a = bq + r$, let $a = r(0)$, $b = r(1)$, $q = q(1)$ and $r = r(2)$. We repeat division as follows:

$a = b*q(1) + r(2), \quad 0 <= r(2) < b$
$b = r(2)q(2) + r(3), \quad 0 <= r(3) < r(2)$
...
$r(i-1) = r(i)q(i) + r(i+1), \quad 0 <= r(i+1) < r(i)$
...
$r(n) = r(n)q(n) + r(n+1), \quad r(n+1) = 0$

$\gcd(a, b) = r(n)$

(5.1) Trace this algorithm with a = 98 and b = 35.

(5.2) Define sequences c and d by

$c(0) = 0, \; c(1) = 1, \; c(i) = c(i-2) - q(i-1)c(i-1)$
$d(0) = 1, \; d(1) = 0, \; d(i) = d(i-2) - q(i-1)d(i-1).$

Then we have $a*d(i) + b*c(i) = r(i)$ for i=0, ..., n. By tracing sequences c and d, compute $5^{-1} \bmod 14$ in the range of $\{1, ..., 13\}$.