Approximation Algorithm of the Knapsack Problem

問題  real numbers $a[1], a[2], \ldots, a[n], w[1], w[2], \ldots, w[n], b$ given
$a[i]$ are profit of item $i$, $w[i]$ are size, and $b$ is the size of knapsack. Let $I$ be a subset of $\{1, 2, \ldots, n\}$. Find $I$ that gives maximum to

$$\text{Max } \sum_{i \in I} a[i] \quad \text{subject to } \sum_{i \in I} w[i] \leq b$$

We maximize the total profit while the total size is limited by $b$.

アルゴリズム (approximation algorithm)

(1) Sort $a[i]/w[i]$ $i=1, \ldots, n$ in non-increasing order.
    Suppose $a[i]/w[i]$ are sorted. That is, item $1$ is most profitable, etc.

(2) Pick up item $1, 2, \ldots$ while the total size is not greater than $b$, that is,
    \[ w[1] + \cdots + w[k] \leq b \] for some $k$. Let $G = a[1] + \cdots + a[k]$. Greedy solution.

(3) Find the maximum of $a[1], \ldots, a[n]$. Let the maximum be $M$. Safe guard.

(4) Let the approximation solution $S$ be $\max\{G, M\}$.

Theorem. Let $R$ be the real solution. Then $S \geq R/2$. That is, $S$ is as bad as half of real solution in the worst case. For random data, $S$ is very close to $R$.

Proof. In the step (2), we assume we break the size constraint at $k+1$ for the first time. That is, $w[1] + \cdots + w[k] + w[k+1] > b$. $a[i]/w[i]$ is the profit ratio of item $i$.
Observe $a[1] + \cdots + a[k] + a[k+1] \geq R$, since we pack items into the knapsack from the most profitable items and the total size exceeds $b$ while the size for the real solution does not exceed $b$. Also observe $M \geq a[k+1]$. Thus

$$R \leq a[1] + \cdots + a[k] + a[k+1] = G + a[k+1] = G + M \leq S + S = 2S.$$  

Theorem. The time complexity of the algorithm is $O(n \log n)$

Proof. The break-down of the algorithm is given by

(1) $O(n \log n)$, (2) $O(n)$, (3) $O(n)$, (4) $O(1)$.

$G=2+b/2$. $M=b/2$. $S=2+b/2$. $R=b/2+b/2=b$. That is $S$ is close to $R/2$.  